INTEGRAL TRANSFORMS AND VECTOR CALCULUS

Unit – 3

FOURIER SERIES

Objectives:

To introduce

- fourier series representation of a given function with period 2π (or) $2l$
- half range series representation of a given function with period π (or) l.

Syllabus:

Determination of Fourier coefficients (without proof) – Fourier series – even and odd functions – Fourier series in an arbitrary interval– Half-range sine and cosine series.

Outcomes:

Students will be able to

- expand the given function as Fourier series in the interval [c , c+ 2π]
- expand the given function as Fourier series in the interval [c , c+ $2l$]
- expand the given function as Half-range Sine [or] Cosine series in the interval [0,l].
- write the expansions of $\frac{\pi^2}{8}, \frac{\pi^2}{6}, \frac{\pi^2}{12}, \ldots$.

Learning Material

Introduction:

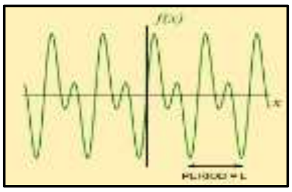
It became important to study the possibility of representation of the given function by infinite series other than power series. Since many phenomena like ***vibration of string, the voltages and currents in electrical networks, electro-magnetic signals, and movement of pendulum are periodic in nature***.

There is a possibility of representing a periodic function as an infinite series involving sinusoidal (sin x & cos x) functions. The French physicist J.B. Fourier announced in his work on heat conduction that an arbitrary periodic function could be expanded in a series of sinusoidal functions.

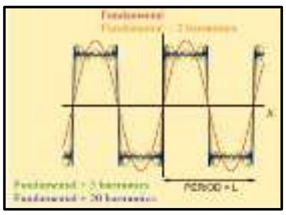
Thus, ***the aim of the theory of Fourier series is to determine the conditions under which the periodic functions can be represented as linear combinations of sine and cosine functions***.

Fourier methods give us a set of powerful tools for representing any periodic function as a sum of sines and cosines.

● A graph of **periodic** function $f(x)$ that has period L exhibits the same pattern every L units along the x-axis, so that $f(x+L) = f(x)$ for every value of x. If we know what the function looks like over one complete period, we can thus sketch a graph of the function over a wider interval of x (that may contain many periods)



One can even approximate a square-wave pattern with a suitable sum that involves a fundamental sine-wave plus a combination of harmonics of this fundamental frequency. This sum is called a **Fourier series**



Existence of Fourier series:

❖ **Dirichlet's Conditions :**

If a function f(x) is defined in $l \le x \le l + 2\pi$, it can be expanded as a Fourier series provided the following Dirichlet's conditions are satisfied

1. **f(x) is singe valued and finite in the interval (c , c + 2 π)**
2. **f(x) is piece-wise continuous with finite number of discontinuities in (c ,c + 2 π).**
3. **f(x) has finite number of maxima or minima in (c ,c + 2 π).**

Note:

- ❖ These conditions are not necessary but only sufficient for the existence of Fourier series.
- ❖ If f(x) satisfies Dirichlet's conditions and f(x) is defined in (c, c + 2 π), then f(x) need not be periodic for the existence of Fourier series of period 2 π .
- ❖ If x = a is a point of discontinuity of f(x), then the value of the Fourier series at x = a is $\frac{1}{2}[f(a+)+f(a-)]$.

Basic Formulae to Solve Integration :

❖ Bracketing Method – [Through Examples]

Continuous Derivatives — Continuous Integration

- $\int x^2 . Sin\, nx\, dx = (x^2)\left(-\frac{Cos\, nx}{n}\right) - (2x)\left(-\frac{Sin\, nx}{n^2}\right) + (2)\left(\frac{Cos\, nx}{n^3}\right)$
- $\int x . Cos\, nx\, dx = (x)\left(\frac{Sin\, nx}{n}\right) - (1)\left(-\frac{Cos\, nx}{n^2}\right)$
- $\int x^2 . Cos\frac{n\pi x}{L}\, dx = (x^2)\left(\frac{Sin\frac{n\pi x}{L}}{\frac{n\pi}{L}}\right) - (2x)\left(-\frac{Cos\frac{n\pi x}{L}}{\frac{n^2\pi^2}{L^2}}\right) + (2)\left(\frac{Cos\frac{n\pi x}{L}}{\frac{n^3\pi^3}{L^3}}\right)$

❖ Spl. Formulae to Remember –

- $\int e^{ax} . Sin\, bx\, dx = \frac{e^{ax}}{a^2+b^2}[a.Sin\, bx - b.Cos bx]$ $\int e^{ax} . Cos bx\, dx = \frac{e^{ax}}{a^2+b^2}[a.Cos bx + b.Sin bx]$
- $\int_{-a}^{a} f(x)\, dx = 2\int_0^a f(x)\, dx$ [Here f(x) must be an **Even** function]
- $\int_{-a}^{a} f(x)\, dx = 0$ [Here f(x) must be an **odd** function]
- Values to Remember : Sin n π = 0 & **Cos n π = $(-1)^n$**

FULL RANGE FOURIER SERIES [Interval of length 2 π]

The Fourier series for the function $f(x)$ in the interval **[c , c + 2 π]** is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

Where $a_0 = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\,dx$, $a_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\cos nx.dx$ & $b_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\sin nx.dx$

C = 0 ➔ $[0, 2\pi]$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

Remember these formulae as this carries 6M Problem.

Where $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$, $a_n = \frac{1}{\pi}\int_0^{2\pi} f(x).Cosnx\,dx$ & $b_n = \frac{1}{\pi}\int_0^{2\pi} f(x).Sin\,nx\,dx$

Remember this formula as this carries 6M Problem.

C = $-\pi$ ➔ $[-\pi, \pi]$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

Where $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$, $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x).Cosnx\,dx$ & $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x).Sin\,nx\,dx$

Examples:

1. Find the Fourier series to represent f(x) = x^2 in the interval $(0,2\pi)$

Sol. As the given interval is $(0, 2\pi)$, Fourier series becomes -

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

Where $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$, $a_n = \frac{1}{\pi}\int_0^{2\pi} f(x).Cosnx\,dx$ & $b_n = \frac{1}{\pi}\int_0^{2\pi} f(x).Sin\,nx\,dx$

<u>Step One</u> : -

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{2\pi} x^2\,dx$$

$$= \frac{1}{\pi}\left[\frac{x^3}{3}\right]_0^{2\pi} = \frac{1}{3\pi}[(2\pi)^3 - 0] = \frac{8}{3}\pi^2$$

$$\Rightarrow a_0 = \frac{8}{3}\pi^2$$

<u>Step Two</u> : -

$$, a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{2\pi} \underbrace{x^2}_{u}\,\underbrace{\cos nx}_{v}\,dx$$

$$= \left[(x^2)\left(\frac{Sin\,nx}{n}\right) - (2x)\left(-\frac{\cos nx}{n^2}\right) + (2)\left(-\frac{Sin\,nx}{n^3}\right)\right]_0^{2\pi}$$

$$= \frac{4}{n^2} \quad \left[\because \begin{matrix}\cos 2n\pi = 1\\ \sin 2n\pi = 0\end{matrix}\right] \quad \Rightarrow a_n = \frac{4}{n^2}$$

<u>Step Three</u> : -

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{2\pi} \underbrace{x^2}_{u}\,\underbrace{\sin nx}_{v}\,d$$

$$= \left[(x^2)\left(-\frac{Cos\,nx}{n}\right) - (2x)\left(-\frac{Sin\,nx}{n^2}\right) + (2)\left(\frac{Cos\,nx}{n^3}\right)\right]_0^{2\pi}$$

$$= -\frac{4\pi}{n} \quad \left[\because \begin{matrix}\cos 2n\pi = 1\\ \sin 2n\pi = 0\end{matrix}\right] \quad \Rightarrow b_n = -\frac{4\pi}{n}$$

Finally, $\therefore f(x) = x^2 = \frac{\frac{8\pi^2}{3}}{2} + \sum_{n=1}^{\infty}\left(\frac{4}{n^2}\cos nx - \frac{4\pi}{n}\sin nx\right)$

$$\Rightarrow x^2 = \frac{4\pi^2}{3} + \sum_{n=1}^{\infty}\left(\frac{4}{n^2}\cos nx - \frac{4\pi}{n}\sin nx\right)$$

2. Express the function f(x) = $\begin{cases} x & 0 < x < \pi \\ \pi & \pi < x < 2\pi \end{cases}$ as Fourier Series .

Sol. As the given interval is (0, 2π), Fourier series becomes -

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

Where $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$, $a_n = \frac{1}{\pi}\int_0^{2\pi} f(x).Cos nx\,dx$ & $b_n = \frac{1}{\pi}\int_0^{2\pi} f(x).Sin\,nx\,dx$

STEP ONE

$$\begin{aligned} a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx &= \frac{1}{\pi}\int_0^{\pi} f(x)\,dx + \frac{1}{\pi}\int_{\pi}^{2\pi} f(x)\,dx \\ &= \frac{1}{\pi}\int_0^{\pi} x\,dx + \frac{1}{\pi}\int_{\pi}^{2\pi} \pi\cdot dx \\ &= \frac{1}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi} + \frac{\pi}{\pi}\Big[x\Big]_{\pi}^{2\pi} \\ &= \frac{1}{\pi}\left(\frac{\pi^2}{2} - 0\right) + \left(2\pi - \pi\right) \\ &= \frac{\pi}{2} + \pi \\ \text{i.e. } a_0 &= \frac{3\pi}{2}. \end{aligned}$$

STEP TWO

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx \\ &= \frac{1}{\pi}\int_0^{\pi} x\cos nx\,dx + \frac{1}{\pi}\int_{\pi}^{2\pi} \pi\cdot\cos nx\,dx \\ &= \underbrace{\frac{1}{\pi}\left[\left[x\frac{\sin nx}{n}\right]_0^{\pi} - \int_0^{\pi}\frac{\sin nx}{n}\,dx\right]}_{\text{using integration by parts}} + \frac{\pi}{\pi}\left[\frac{\sin nx}{n}\right]_{\pi}^{2\pi} \\ &= \frac{1}{\pi}\left[\frac{1}{n}\left(\pi\sin n\pi - 0\cdot\sin n0\right) - \left[\frac{-\cos nx}{n^2}\right]_0^{\pi}\right] \\ &\quad + \frac{1}{n}(\sin n2\pi - \sin n\pi) \end{aligned}$$

$$\begin{aligned} \text{i.e. } a_n &= \frac{1}{\pi}\left[\frac{1}{n}\left(0-0\right) + \left(\frac{\cos n\pi}{n^2} - \frac{\cos 0}{n^2}\right)\right] + \frac{1}{n}\left(0-0\right) \\ &= \frac{1}{n^2\pi}(\cos n\pi - 1), \quad \text{see TRIG} \\ &= \frac{1}{n^2\pi}((-1)^n - 1). \end{aligned}$$

$$\text{i.e. } a_n = \begin{cases} -\frac{2}{n^2\pi} & , n \text{ odd} \\ 0 & , n \text{ even.} \end{cases}$$

STEP THREE

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx \\ &= \frac{1}{\pi}\int_0^{\pi} x\sin nx\,dx + \frac{1}{\pi}\int_{\pi}^{2\pi} \pi\cdot\sin nx\,dx \\ &= \underbrace{\frac{1}{\pi}\left[\left[x\left(-\frac{\cos nx}{n}\right)\right]_0^{\pi} - \int_0^{\pi}\left(\frac{-\cos nx}{n}\right)dx\right]}_{\text{using integration by parts}} + \frac{\pi}{\pi}\left[\frac{-\cos nx}{n}\right]_{\pi}^{2\pi} \\ &= \frac{1}{\pi}\left[\left(\frac{-\pi\cos n\pi}{n} + 0\right) + \left[\frac{\sin nx}{n^2}\right]_0^{\pi}\right] - \frac{1}{n}(\cos 2n\pi - \cos n\pi) \\ &= \frac{1}{\pi}\left[\frac{-\pi(-1)^n}{n} + \left(\frac{\sin n\pi - \sin 0}{n^2}\right)\right] - \frac{1}{n}(1-(-1)^n) \\ &= -\frac{1}{n}(-1)^n + \quad 0 \quad - \frac{1}{n}(1-(-1)^n) \end{aligned}$$

Hence the Fourier series becomes ,

$$f(x) = \frac{1}{2}\left(\frac{3\pi}{2}\right) + \left(-\frac{2}{\pi}\right)\left[\cos x + 0\cdot\cos 2x + \frac{1}{3^2}\cos 3x + \dots\right]$$

$$+ \left(-1\right)\left[\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \dots\right]$$

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3. **Express $f(x) = x - \pi$ as Fourier series in the interval $-\pi < x < \pi$**

Sol Let the function $x-\pi$ be represented by the Fourier series

$$x - \pi = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \rightarrow (1)$$

Then

Sep – 1

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x-\pi)dx$$
$$= \frac{1}{\pi}\left[\int_{-\pi}^{\pi} xdx - \pi\int_{-\pi}^{\pi} dx\right]$$
$$= \frac{1}{\pi}\left[0 - \pi.2\int_{0}^{\pi} dx\right] \quad (\because x \text{ is odd function})$$
$$= \frac{1}{\pi}\left[-2\pi\,(x)_0^{\pi}\right]$$
$$= -2(\pi - 0) = -2\pi \text{ and}$$

Step - 2

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx.dx$$
$$= \frac{1}{\pi}\int_{-\pi}^{\pi}(x-\pi)\cos nx.dx$$
$$= \frac{1}{x}\left[\int_{-\pi}^{\pi} x\cos nx.dx - \pi\int_{-\pi}^{\pi}\cos nx.dx\right]$$
$$= \frac{1}{\pi}\left[0 - 2\pi\int_{0}^{\pi}\cos nx.dx\right]$$

($\because x\cos nx$ is odd function and $\cos nx$ is even function)

$$\therefore a_n = -2\int_0^{\pi}\cos nx.dx$$
$$= -2\left(\frac{\sin nx}{n}\right)_0^{\pi}$$
$$= \frac{-2}{n}(\sin n\pi - \sin 0)$$
$$= \frac{-2}{n}(0-0) = 0 \quad for \ n = 1,2,3\dots\dots$$

Sep – 3

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx.dx$$
$$= \frac{1}{\pi}\int_{-\pi}^{\pi}(x-\pi)\sin nx.dx$$
$$= \frac{1}{\pi}\left[\int_{-\pi}^{\pi} x\sin nx - \pi\int_{-\pi}^{\pi}\sin nx.dx\right]$$
$$= \frac{1}{\pi}\left[2\int_0^{\pi} x\sin nx.dx - \pi\,(0)\right]$$
$$= \frac{2}{\pi}\left[x\left(\frac{-\cos nx}{n}\right) - 1\left(\frac{-\sin nx}{n^2}\right)\right]_0^{\pi}$$
$$= \frac{2}{\pi}\left[\left(\frac{-\pi\cos n\pi}{n} + 0\right) - (0+0)\right]$$
$$= \frac{-2}{\pi}\cos n\pi = \frac{-2}{n}(-1)^n$$
$$= \frac{2}{n}(-1)^{n+1}\ \forall n = 1,2,3\dots..$$

Substituting the values of a_0, a_n, b_n in (1),

We get ,

$$x - \pi = -\pi + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2}{\pi} \sin nx$$
$$= -\pi + 2\left[\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \frac{1}{4}\sin 4x + \dots\dots\right]$$

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4. Find the Fourier Series of the periodic function defined as $f(x) = \begin{cases} -\pi & -\pi < x < 0 \\ x & 0 < x < \pi \end{cases}$

   Hence deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \text{------} = \frac{\pi^2}{8}$

Sol. Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \rightarrow (1)$ Then

Step 1 :

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$$
$$= \frac{1}{\pi}\left[\int_{-\pi}^{o} (-\pi)dx + \int_0^{\pi} xdx\right]$$
$$= \frac{1}{\pi}\left[-\pi (x)_{-\pi}^{0} + \left(\frac{x^2}{2}\right)_0^{\pi}\right]$$
$$= \frac{1}{\pi}\left[-\pi^2 + \frac{\pi^2}{2}\right] = \frac{1}{\pi}\left[\frac{-\pi^2}{2}\right] = \frac{-\pi}{2}$$

Step 2 :

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx.dx$$
$$= \frac{1}{\pi}\left[\int_{-\pi}^{o} (-\pi)\cos nx.dx + \int_0^{\pi} x\cos nx.dx\right]$$
$$= \frac{1}{\pi}\left[-\pi\left(\frac{\sin nx}{n}\right)_{-\pi}^{0} + \left(x\frac{\sin nx}{n} + \frac{\cos nx}{n^2}\right)_0^{\pi}\right]$$
$$= \frac{1}{\pi}\left[0 + \frac{1}{n^2}\cos n\pi - \frac{1}{\pi n^2}\right]$$

$$= \frac{1}{\pi}\left[\frac{(-1)^n}{n^2} + \frac{1}{n^2}\right] = \frac{1}{\pi n^2}[(-1)^n - 1]$$

$$\rightarrow a_n = \frac{1}{\ }[(-1)^n - 1]$$

Step 3 :

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx.dx$$
$$= \frac{1}{\pi}\left[\int_{\pi}^{0} (-\pi)\sin nx.dx + \int_0^{\pi} x\sin nx.dx\right]$$
$$= \frac{1}{\pi}\left[\pi\left(\frac{\cos nx}{n}\right)_{-\pi}^{0} + \left(-x\frac{\cos nx}{n} + \frac{\sin nx}{n^2}\right)_0^{\pi}\right]$$
$$= \frac{1}{\pi}\left[\frac{\pi}{n}(1 - \cos n\pi) - \frac{\pi}{n}\cos n\pi\right]$$
$$= \frac{1}{n}(1 - 2\cos n\pi)$$

$$b_1 = 3, b_2 = \frac{-1}{2}, b_3 = 1, b_4 = \frac{-1}{4} \dots\dots\dots$$

Substituting the values of a_0, a_n *and* b_n in (1), we get

$$f(x) = \frac{-\pi}{4} - \frac{2}{\pi}\left(\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \text{---}\right) + \left(3\sin x - \frac{\sin 2x}{2} + \frac{3\sin 3x}{3} - \frac{\sin 4x}{4} + \text{--}\right)$$

Deduction: Put $x = 0$ in the above function $f(x)$, we get

$$f(0) = -\frac{\pi}{\ } - \frac{2}{\ }\left(1 + \frac{1}{\ } + \frac{1}{\ } + \dots\right)$$

Even and Odd Functions:-

A function $f(x)$ is said to be even $if \ f(-x)=f(x)$ and odd if $f(-x)=-f(x)$

Example :- $x^2, x^4+x^2+1, e^x+e^{-x}$ are even functions & $x^3, x, \sin x, \cos ecx$ are odd functions.

Note 1 :-

1. Product of two even (or) two odd functions will be an even function
2. Product of an even function and an odd function will be an odd function

Note 2:- $\int_{-a}^{a} f(x)dx = 0$ when $f(x)$ is an odd function

$= 2\int_{0}^{a} f(x)dx$ When $f(x)$ is even function

Fourier series for even and odd functions

We know that a function $f(x)$ defined in $(-\pi,\pi)$ can be represented by the Fourier series

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos nx+\sum_{n=1}^{\infty} b_n \sin nx$$

Where $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$, $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx.dx$

And $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx.dx$

Case 1 :- when $f(x)$ is **even function**

Since $\cos nx$ is an even function,

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)dx$$

➔ $f(x)\cos nx$ is also an even function.

$a_n = \frac{1}{\pi}\int^{\pi} f(x)\cos nx.dx$

Case 2:- when $f(x)$ is an **odd function**

since $f(x)$ is an odd function

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = 0$$

➔ Since $\cos nx$ is an even function, $f(x)\cos nx$ is an odd function and

Examples:-

1. Expand the function $f(x) = x^2$ as a Fourier series in $(-\pi, \pi)$, hence deduce that

$$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + - - - - - = \frac{\pi^2}{12}$$

Sol. Since $f(-x) = (-x)^2 = x^2 = f(x)$ ➔ ***f(x) is an even function.***

Hence in its Fourier series expansion, the sine terms are absent

$$\therefore\ x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$\therefore\ b_n = 0$

No need to find

Step 1 :

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x^2 dx$$

$$= \frac{2}{\pi}\left(\frac{x^3}{3}\right)_0^{\pi} = \frac{2\pi^2}{3}$$

Step 2 :

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx.dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx.dx$$

$$= \frac{2}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) - 2x\left(\frac{-\cos nx}{n^2}\right) + 2\left(\frac{-\sin nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[0 + 2\pi\frac{\cos nx}{n^2} + 2.0\right]$$

$$= \frac{4\cos n\pi}{n^2} = \frac{4}{n^2}(-1)^n$$

Substituting the values of a_0 *and* a_n , we get

$$x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2}(-1)^n \cos nx$$

$$= \frac{\pi^2}{3} - 4\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} \cos nx$$

$$= \frac{\pi^2}{3} - 4\left(\cos x - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \frac{\cos 4x}{4^2} + ---\right)$$

Deductions:- Putting $x = 0$ in (4), we get

$$0 = \frac{\pi^2}{3} - 4\left(1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + ---\right)$$

$$\Rightarrow 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + ---- = \frac{\pi^2}{12}$$

FULL RANGE FOURIER SERIES *[Interval of length 2l]*

The Fourier series for the function $f(x)$ in the interval **[c, c + 2*l*]** is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n Cos\frac{n\pi x}{l} + b_n Sin\frac{n\pi x}{l}\right)$$

Where $a_0 = \frac{1}{l}\int_c^{c+2l} f(x)\,dx$, $a_n = \frac{1}{l}\int_c^{c+2l} f(x) Cos\frac{n\pi x}{l}\,dx$, $b_n = \frac{1}{l}\int_c^{c+2l} f(x)\, Sin\frac{n\pi x}{l}\,dx$

C = 0 ➔ **[0, 2*l*]**

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n Cos\frac{n\pi x}{l} + b_n Sin\frac{n\pi x}{l}\right)$$

Remember this formula as this carries 6M Problem.

Where $a_0 = \frac{1}{l}\int_0^{2l} f(x)\,dx$, $a_n = \frac{1}{l}\int_0^{2l} f(x) Cos\frac{n\pi x}{l}\,dx$ & $b_n = \frac{1}{l}\int_0^{2l} f(x)\, Sin\frac{n\pi x}{l}\,dx$

C = −*l* ➔ **[−*l*, *l*]**

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n Cos\frac{n\pi x}{l} + b_n Sin\frac{n\pi x}{l}\right)$$

Remember this formula as this carries 6M Problem.

Where $a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx$, $a_n = \frac{1}{l}\int_{-l}^{l} f(x) Cos\frac{n\pi x}{l}\,dx$ & $b_n = \frac{1}{l}\int_{-l}^{l} f(x)\, Sin\frac{n\pi x}{l}\,dx$

Examples:-

1. **Express $f(x) = x^2$ as a Fourier series in $[-l, l]$**

SEE FOR ***EVEN*** OR ***ODD*** FUNCTION AS THE INTERVAL IS FROM – VALUE TO + VALUE

Sol $f(-x) = f(-x)^2 = x^2 = f(x)$

Therefore $f(x)$ is an even function

Hence the Fourier series of $f(x)$ in $[-l, l]$ is given by

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$$

$$where\ a_n=\frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}dx$$

$$hence\ a_0=\frac{2}{l}\int_0^l x^2\,dx=\frac{2}{l}\left(\frac{x^3}{3}\right)_0^l=\frac{2l}{3}$$

$$also\ a_n=\frac{2}{l}\int_0^l x^2\cos\frac{n\pi x}{l}dx$$

$$=\frac{2}{l}\left[x^2\left[\frac{\sin\left(\frac{n\pi x}{l}\right)}{\frac{n\pi}{l}}\right]-2x\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)+2\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}}\right)\right]_0^l$$

$$=\frac{2}{l}\left[2x\,\frac{\cos\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right]_0^l$$

Since the first and last terms vanish at both upper and lower limits

$$\therefore a_n=\frac{2}{l}\left[2l\,\frac{\cos n\pi}{n^2\pi^2/l^2}\right]=\frac{4l^2\cos n\pi}{n^2\pi^2}$$

$$=\frac{(-1)^n 4l^2}{n^2\pi^2}$$

Substituting these values in (1), we get

$$x^2=\frac{l^2}{3}+\sum_{n=1}^{\infty}\frac{(-1)^n 4l^2}{n^2\pi^2}\cos\frac{n\pi x}{l}$$

$$=\frac{l^2}{3}-\frac{4l^2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\cos\frac{n\pi x}{l}$$

$$=\frac{l^2}{3}-\frac{4l^2}{\pi^2}\left[\frac{\cos(\pi x/l)}{1^2}-\frac{\cos(2\pi x/l)}{2^2}+\frac{\cos(3\pi x/l)}{3^2}-----\right]$$

---

2. Find a Fourier series with period 3 to represent $f(x)=x+x^2\ in\ (0,3)$

Sol. Let $$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{l}+b_n\sin\frac{n\pi x}{l}\right)\to(1)$$

Here $2l=3,\quad l=3/2$. Hence (1) becomes

$$f(x)=x+x^2$$

$$=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\frac{2n\pi x}{3}+b_n\sin\frac{2n\pi x}{3}\right)\to(2)$$

$$Where\ a_0=\frac{1}{l}\int_0^{2l} f(x)dx$$

$$=\frac{2}{3}\int_0^3(x+x^2)dx=\frac{2}{3}\left[\frac{x^2}{2}+\frac{x^3}{3}\right]_0^3=9$$

$$and\ a_n=\frac{1}{l}\int_0^2 f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$=\frac{2}{3}\int_0^3(x+x^2)\cos\left(\frac{2n\pi x}{3}\right)dx$$

Using bracketing method, we obtain

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}\,dx \qquad = \frac{2}{3}\int_0^{3}\left(x + x^2\right)\sin\left(\frac{2n\pi x}{3}\right)dx$$

Substituting the values of a's and b's in (2) we get

$$x + x^2 = \frac{9}{2} + \frac{9}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos\left(\frac{2n\pi x}{3}\right) - \frac{12}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{2n\pi x}{3}\right)$$

Half –Range Fourier Series (Interval of length l) ➔ [0,l]

Remember these formulae as this carries 6M Problem.

Part – B [3Q – (b)]

The Cosine series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n.Cos\frac{n\pi x}{l}$$

Where

$$a_0 = \frac{2}{l}\int_0^{l} f(x)\,dx$$

$$a_n = \frac{2}{l}\int_0^{l} f(x).Cos\frac{n\pi x}{l}$$

The sine series

$$f(x) = \sum_{n=1}^{\infty} b_n.Sin\frac{n\pi x}{l}$$

Where

$$b_n = \frac{2}{l}\int_0^{l} f(x).Sin\frac{n\pi x}{l}$$

Note:-

1) Suppose $f(x) = x \ in \ [0,\pi]$, it can have Fourier cosine series expansion as well as Fourier sine series expansion in $[0,\pi]$
2) If $f(x) = x^2 \ in \ [0,\pi]$, can have Fourier cosine series as well as sine series

Examples:-

1. Find the **half range sine** series for $f(x) = x(\pi - x) \ in \ 0 < x < \pi$. Deduce that $\frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + - - - - = \frac{\pi^3}{32}$

Ans. The ***Fourier sine series*** expansion of $f(x)$ in $(0,\pi)$ is

Half range ➔

(0,l) means (0,π)

$$f(x) = x(\pi - x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$where\ b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx.dx$$

$$hence\ b_n = \frac{2}{\pi}\int_0^{\pi} x(\pi - x)\sin nx.dx = \frac{2}{\pi}\int_0^{\pi}(\pi x - x^2)\sin nx.dx$$

$$= \frac{2}{\pi}\left[(\pi x - x^2)\left(\frac{-\cos nx}{n}\right) - (\pi - 2x)\left(\frac{-\sin nx}{n^2}\right) + (-2)\frac{\cos nx}{n^3}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{2}{n^3}(1 - \cos n\pi)\right]$$

$$= \frac{4}{n\pi^3}\left(1 - (-1)^n\right)$$

$$b_n = \begin{cases} 0, & when\ n\ is\ even \\ \frac{8}{\pi n^3}, & when\ n\ is\ odd \end{cases}$$

Hence

$$x(\pi - x) = \sum_{n=1,3,5...} \frac{8}{\pi n^3}\sin nx \ (or)$$

$$x(\pi - x) = \frac{8}{\pi}\left(\sin x + \frac{\sin 3x}{3^3} + \frac{\sin 5x}{5^3} + ----\right) \to (1)$$

Deduction:- Putting $x = \frac{\pi}{2}$ in (1), we get

$$\frac{\pi}{2}\left(x - \frac{\pi}{2}\right) = \frac{8}{\pi}\left(\sin\frac{\pi}{2} + \frac{1}{3^3}\sin\frac{3\pi}{2} + \frac{1}{5^3} sin\frac{5\pi}{2} + --\right)$$

$$\frac{\pi^2}{4} = \frac{8}{\pi}\left[1 + \frac{1}{3^3}\sin\left(\pi + \frac{\pi}{2}\right) + \frac{1}{5^3}\sin\left(2\pi + \frac{\pi}{2}\right) + \frac{1}{7^3}\sin\left(3\pi + \frac{\pi}{2}\right) + ---\right]$$

Hence

$$\frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + ---- = \frac{\pi^3}{32}$$

---

2. Find the **half- range sine** series of $f(x) = 1\ in\ [0,l]$

Ans. The Fourier sine series of $f(x)\ in\ [0,l]$ is given by $f(x) = 1 = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$

$$here\ b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx$$

$$= \frac{2}{l}\int_0^l 1.\sin\frac{n\pi x}{l}dx$$

$$= \frac{2}{l}\left(\frac{-\cos\frac{n\pi x}{l}}{n\pi / l}\right)_0^l$$

$$= \frac{2}{n\pi}\left[-\cos\frac{n\pi x}{l}\right]_0^l$$

$$= \frac{2}{n\pi}\left(-\cos n\pi + 1\right)$$

$$= \frac{2}{n\pi}\left[(-1)^{n+1} + 1\right]$$

$\therefore b_n = 0$ when n is even

$= \frac{4}{n\pi}$, when n is odd

Hence the required Fourier series is $f(x) = \sum_{n=1,3,5---}^{\infty} \frac{4}{n\pi}\sin\frac{n\pi x}{l}$.

3. Find the **half – range cosine** series expansion of $f(x) = \sin\left(\frac{\pi x}{l}\right)$ in the range $0 < x < l$

Sol. Half Range Cosine series in (0,l) is given by $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n.Cos\frac{n\pi x}{l}$

$$where\ a_0 = \frac{2}{l}\int_0^l f(x)dx = \frac{2}{l}\int_0^l \sin\frac{\pi x}{l}dx$$

$$= \frac{2}{l}\left[\frac{-\cos\pi x/l}{\pi/l}\right]_0^l$$

$$= \frac{2}{l}(\cos\pi - 1) = \frac{4}{\pi} and$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}dx$$

$$= \frac{2}{l}\int_0^l \sin\left(\frac{\pi x}{l}\right)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{1}{l}\int_0^l\left[\frac{\sin(n+1)\pi x}{l} - \frac{\sin(n-1)\pi x}{l}\right]dx$$

$$= \frac{1}{l}\left[-\frac{\frac{\cos(n+1)\pi x}{l}}{(n+1)\pi/l} + \frac{\cos(n-1)\pi x/l}{(n-1)\pi/l}\right]_0^l$$

$$= \frac{1}{\pi}\left[-\frac{(-1)^{n+1}}{n+1} + \frac{(-1)^{n-1}}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right]$$

When n is odd

$$a_n = \frac{1}{\pi}\left[\frac{-1}{n+1} + \frac{1}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right] = 0$$

When n is even

$$a_n = \frac{1}{\pi}\left[\frac{1}{n+1} - \frac{1}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right]$$

$$= \frac{-4}{\pi\,(n+1)(n-1)}$$

$$\therefore \sin\left(\frac{\pi x}{l}\right) = \frac{2}{\pi} - \frac{4}{\pi}\left[\frac{\cos(2\pi x/l)}{1.3} + \frac{\cos(4\pi x/l)}{3.5} + ----\right]$$

=======*========**=======*========**=======*========**=======*========

Assignment-Cum-Tutorial Questions

SECTION-A

Objective Questions

1. If $f(x)=\begin{cases}1+\frac{2x}{\pi} & -\pi \le x \le 0 \\ 1-\frac{2x}{\pi} & 0 \le x \le \pi\end{cases}$ Then $f(x)$ is__________function [ ]

   a) Odd b) even c) periodic d) none

2. If the Fourier series for the function $f(x)$ defined in $[-\pi,\pi]$ then a_n = __________

3. The Fourier constant b_n *for* $f(x)=x\sin x$ *in* $[-\pi,\pi]$ is _______

4. If $f(x)=x^2$ in $(-l,l)$ then a_0 & b_1 are ____________

5. If $f(x)=|x|$ in $(-\pi,\pi)$ then a_1 & b_1 *are* _______________

6. In Fourier expansion of $f(x)=x+x^2$ in $(-\pi,\pi)$ the value of a_n is [ ]

   a) $\frac{2}{n^2}(-1)^4$ b) $\frac{4}{n^2}(-1)^n$ c) 0 d) none

7. If $f(x)=x\cos x$ *in* $(-\pi,\pi)$ then a_n is [ ]

   a) 1 b) 2 c) 3 d) 0

8. If $f(x)$ is expanded as a Fourier series in $(0,2\pi)$ then a_0 = _____________ [ ]

   a) $\frac{1}{\pi}\int_0^{2\pi} f(x)dx$ b) $\frac{1}{\pi}\int_0^{\pi} f(x)dx$ c) $\frac{2}{\pi}\int_0^{2\pi} f(x)dx$ d) none

9. Fourier sine series for $f(x)=x$ in $(0,\pi)$ is ____________________

10. If $f(x)=\ sin\,x$ in $-\pi<x<\pi$ then a_0 =_________________

11. In Fourier series expansion of $f(x)=coshx$ in $(-4,4)$ the Fourier co efficient a_1 is ____________________

12. If $f(x)$ is expanded as a Fourier series in $[0,2\pi]$ then b_n = __________

    [ ]

    a) $\frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx.dx$ b) $\frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx.dx$

    c) $\frac{2}{\pi}\int_0^{2\pi} f(x)\sin nx.dx$ d) none

13. 10. If $f(x)=1+\sin x$ in $(-1,1)$ is expressed as a Fourier series then the Value of b_n =_______[ ]

    a) 0 b) 1 c) 2 d) none

SECTION-B

II) Level Two Questions:

1. Obtain Fourier Series for the function $f(x) = \begin{cases} x, & if\ 0 < x < \pi \\ 2\pi - x, & if\ \pi < x < 2\pi \end{cases}$

   And hence deduce that $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots$

2. Obtain the Fourier series to represent $x - x^2$ in ($-\pi$, π) and deduce that

   $\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \dots$

3. If $f(x) = x^2$, $-l < x < l$. Obtian Fourier Series and deduce that $\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \dots$

4. Expand $f(x) = e^{-x}$ as a Fourier series in$(-1,1)$.

5. Obtain Fourier series to represent the function $f(x) = |x|$ in ($-\pi$,π)and deduce that

   $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots$

6. Obtain the Fourier series expansion of f(x) given that $f(x) = (\pi - x)^2\ in\ 0 < x < 2\pi$ and deduce that $1/1^2 + 1/2^2 + 1/3^2 + \dots\dots\dots\dots = \pi^2/6$

7. Find a Fourier series to represent the function $f(x) = e^x$ for $-\pi < x < \pi$ and hence derive a series for $\pi/sinh\pi$

8. Find the Fourier series of the periodic function $f(x) = \begin{cases} -\pi\ , -\pi < x < \pi \\ x\ ,\ 0 < x < \pi \end{cases}$

   Hence deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots \dots\dots\dots. = \frac{\pi^2}{8}$

9. Find the half-range cosine series and sine series for $f(x) = x$ in $0 < x < \pi$ hence deduce that

   $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots \dots\dots\dots = \frac{\pi^2}{8}$

10. Find the Fourier series expansion for $f(x) = \begin{cases} 2, -2 < x < 0 \\ x, 0 < x < 2 \end{cases}$

11. Find the Fourier series expansion for the function $f(x) = x - x^2\ in\ (-1,1)$

12. Show that the Fourier series expansion of f(x) = 1 in 0<x<1 and f(x) = 2 in 1<x<3 with f(x+3) = f(x) is $\frac{5}{3} + \frac{9}{4\pi}\left[\frac{\sqrt{3}}{2}\cos\left(\frac{3\pi x}{2}\right) - \frac{\sqrt{3}}{4}\cos 3\pi x + \dots\right] + \frac{9}{4\pi}\left[-\frac{3}{2}\sin\left(\frac{3\pi x}{2}\right) - \frac{3}{4}\sin 3\pi x + \dots\right]$ (DEC 2015)

13. Find the half-range cosine series for the function $f(x) = \begin{cases} kx\ , 0 \le x \le \frac{l}{2} \\ k(l - x), \frac{l}{2} \le x \le l \end{cases}$

14. Express f(x) = x as a half range sine series in 0 < x < 2.

15. Find the half-range cosine series for the function $f(x) = (x - 1)^2$ in the interval $0 < x < 1$

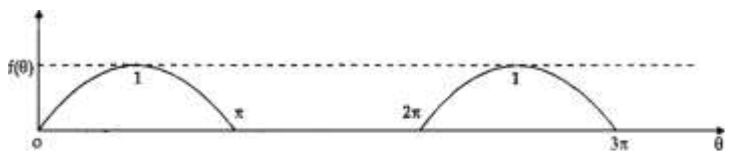
    Hence show that $\sum_{n=1}^{\infty} \frac{1}{(2x-1)^2} = \frac{\pi^2}{8}$

SECTION-C

C. Questions testing the analyzing / evaluating ability of students

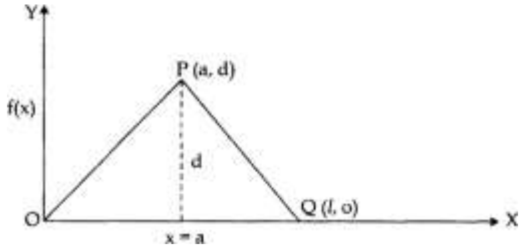
Level Three Questions:

1. An alternating current after passing through a rectifier has form $i = \begin{cases} I.\sin\theta & 0 < \theta < \pi \\ 0 & \pi < \theta < 2\pi \end{cases}$.



Find the Fourier series of the function.

2. Find the half period series for f(x) given in the range (0,L) by the graph OPQ as shown in the following fig.



Hint. $f(x) = \begin{cases} \dfrac{xd}{a}, & o < x < a \\ \dfrac{d(l - x)}{l - a}, & a < x < l \end{cases}$.

Gate Previous year Questions :

2016

Let $f(x)$ be a real, periodic function satisfying $f(-x) = -f(x)$. The general form of its Fourier series representation would be

(A) $f(x) = a_0 + \sum_{k=1}^{\infty} a_k \cos(kx)$

(B) $f(x) = \sum_{k=1}^{\infty} b_k \sin(kx)$

(C) $f(x) = a_0 + \sum_{k=1}^{\infty} a_{2k} \cos(kx)$

(D) $f(x) = \sum_{k=0}^{\infty} a_{2k+1} \sin(2k+1)x$

2015

The signum function is given by

$$sgn(x) = \begin{cases} \dfrac{x}{|x|}; x \neq 0 \\ 0; x = 0 \end{cases}$$

The Fourier series expansion of $sgn(\cos(t))$ has

(A) only sine terms with all harmonics.
(B) only cosine terms with all harmonics.
(C) only sine terms with even numbered harmonics.
(D) only cosine terms with odd numbered harmonics.

Options :

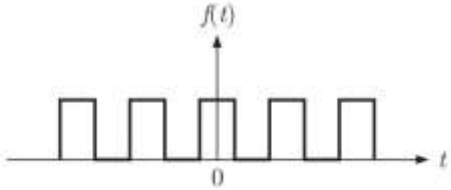
1. ✖ A
2. ✖ B
3. ✖ C
4. ✔ D

2012 Let $x(t)$ be a periodic signal with time period T, Let $y(t) = x(t - t_0) + x(t + t_0)$ for some t_0. The Fourier Series coefficients of $y(t)$ are denoted by b_k. If $b_k = 0$ for all odd k, then t_0 can be equal to

(A) $T/8$ (B) $T/4$

(C) $T/2$ (D) $2T$

2011 The Fourier series expansion $f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega t + b_n \sin n\omega t$ of the periodic signal shown below will contain the following nonzero terms



(A) a_0 and $b_n, n = 1,3,5,...\infty$ (B) a_0 and $a_n, n = 1,2,3,...\infty$

(C) $a_0\ a_n$ and $b_n, n = 1,2,3,...\infty$ (D) a_0 and $a_n\ n = 1,3,5,...\infty$

2010 The period of the signal $x(t) = 8\sin\left(0.8\pi t + \frac{\pi}{4}\right)$ is

(A) 0.4π s (B) 0.8π s

(C) 1.25 s (D) 2.5 s

2009

The Fourier Series coefficients of a periodic signal $x(t)$ expressed as $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi kt/T}$ are given by $a_{-2} = 2 - j1$, $a_{-1} = 0.5 + j0.2$, $a_0 = j2$, $a_1 = 0.5 - j0.2$, $a_2 = 2 + j1$ and $a_k = 0$ for $|k| > 2$

Which of the following is true ?

(A) $x(t)$ has finite energy because only finitely many coefficients are non-zero

(B) $x(t)$ has zero average value because it is periodic

(C) The imaginary part of $x(t)$ is constant

(D) The real part of $x(t)$ is even

2008

Let $x(t)$ be a periodic signal with time period T, Let $y(t) = x(t - t_0) + x(t + t_0)$ for some t_0. The Fourier Series coefficients of $y(t)$ are denoted by b_k. If $b_k = 0$ for all odd k, then t_0 can be equal to

(A) $T/8$ (B) $T/4$

(C) $T/2$ (D) $2T$

2007

A signal $x(t)$ is given by

$$x(t) = \begin{cases} 1, -T/4 < t \le 3T/4 \\ -1, 3T/4 < t \le 7T/4 \\ -x(t+T) \end{cases}$$

Which among the following gives the fundamental fourier term of $x(t)$?

(A) $\frac{4}{\pi}\cos\left(\frac{\pi t}{T} - \frac{\pi}{4}\right)$ (B) $\frac{\pi}{4}\cos\left(\frac{\pi t}{2T} + \frac{\pi}{4}\right)$

(C) $\frac{4}{\pi}\sin\left(\frac{\pi t}{T} - \frac{\pi}{4}\right)$ (D) $\frac{\pi}{4}\sin\left(\frac{\pi t}{2T} + \frac{\pi}{4}\right)$

****

UNIT - 1

MATRICES

- **Rank of a matrix: Let** A is be an matrix .If A is null matrix , we define its rank to be 0 (zero).
- If A is non zero matrix ,we say that ‘r’ is the rank of A if
- (i) every (r+1)th order minor of A is 0(zero) and
- (ii) there exists at least one rth order minor of A which is not zero
- Rank of A is denoted by ρ(A)
- **Note:**
- **1) E**very matrix will have rank
- 2) Rank of a matrix is unique
- 3) ρ (A) 1 when A is a non-zero matrix
- 4)If A is a matrix of order rank of A= ρ (A) min(m,n)
- 5)If ρ (A) = r then every minor of A of order r+1 or more is zero
- 6)Rank of the identity matrix In is n
- 7)If A is a matrix of order ‘n’ and A is non-singular (i.e; det A 0) then ρ (A)=n.
- 8)The rank of the transpose of a matrix is the some as that of the original matrix(i.e; ρ (A)= ρ (AT))
- 9) If A and B are two equivalent matrices then rank A= rank B
- 10)if A and B are two equivalent matrixes then rank A = rank B.

Find the rank of the matrix $A = \begin{bmatrix} -1 & 0 & 6 \\ 3 & 6 & 1 \\ -5 & 1 & 3 \end{bmatrix}_{3x3}$

ol: Det A of given matrix (A) = -1(18-1) – 0(9+5)+(3+30) = -17-0+198

= 181≠0

A is non – singular third order matrix

rank of A = ρ (A) = 3 = order of given matrix.

2) Find rank of the matrix $\begin{bmatrix} 1 & -2 & -1 \\ -3 & 3 & 0 \\ 2 & 2 & 4 \end{bmatrix}$

Sol:- det A = (A) = 1(12-0) – (-2) (-12-0) -1(-6-6)

= 12-24+12=0

∴ A is singular

Let us take a submatrix of given matrix

$B=\begin{bmatrix} 1 & -2 \\ -3 & 3 \end{bmatrix} \Rightarrow \{B\} = 3-6 = -3 \neq 0$

Rank of given matrix = submatrix rank = P(A) = 2

Echelon form:-

The Echelon form of a matrix A is an equivalent matrix, obtained by finite number of elementary operations on A by the following way.

1) The zero rows, if any, are below a nonzero row

2) The first nonzero entry in each nonzero row is one (1)

3) The number of zeros before the first nonzero entry in a row is lessthan the number of such zeros in the next row immediately below it.

Note:- (i) Condition (2) is optinal

(ii) The rank of A is equal to the number of nonzero rows in its echelon form.

Solved Problems:

1) Find the rank of the matrix by echelon form $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}$

Sol:- Given A $= \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}$

$R_2 \rightarrow R_2 - R_1$; $R_3 \rightarrow R_3 - 2R_1$

$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -1 \\ 0 & 2 & -1 \end{bmatrix}$

$R_3 \rightarrow R_3 - 2R_3$

$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix}$

$\therefore$ ρ (A) = Rank of A = number of non zero rows = 2

2) Find the rank of the matrix $\begin{bmatrix} 4 & 2 & 3 \\ 8 & 4 & 6 \\ -2 & -1 & -15 \end{bmatrix}$

Sol :- Given A = $\begin{bmatrix} 4 & 2 & 3 \\ 8 & 4 & 6 \\ -2 & -1 & -15 \end{bmatrix}$

$R_2 \to R_2 - 2R_1$; $R_3 \to 2R_3 + R_1$

$\sim \begin{bmatrix} 4 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\therefore$ Rank of A = ρ (A) = Number of non zero rows = 1

3) Find the value of K such that the rank of A = $\begin{bmatrix} 1 & 1 & -1 & 1 \\ 1 & -1 & k & -1 \\ 3 & 1 & 0 & 1 \end{bmatrix}$ is 2

Sol:- Given A = $\begin{bmatrix} 1 & 1 & -1 & 1 \\ 1 & -1 & k & -1 \\ 3 & 1 & 0 & 1 \end{bmatrix}$

$R_2 \to R_2 - R_1$; $R_3 \to R_3 - 3R_1$

$\sim \begin{bmatrix} 1 & +1 & -1 & 1 \\ 0 & -2 & k+1 & -2 \\ 0 & -2 & +3 & -2 \end{bmatrix}$

$R_3 \to R_3 - R_2$

$\sim \begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & -2 & k+1 & -2 \\ 0 & 0 & -k+2 & 0 \end{bmatrix}$

Give rank of A is 2, there will be only two non zero rows

$\therefore$ Third row must be zero row $\Rightarrow$ 2-K=0

$\Rightarrow$ K = 2

Normal form:

Every m x n matrix of rank r can be reduced to the for [Ir 0] or Ir or (3) $\begin{bmatrix} Ir & 0 \\ 0 & 0 \end{bmatrix}$ by a finale number of elementary row or column transformations. Here 'r' indicates rank of the matrix.

Solved Problems:

1) Find the rank of the matrix by using normal form where A= $\begin{bmatrix} 2 & 3 & 7 \\ 3 & -2 & 4 \\ 1 & -3 & -1 \end{bmatrix}$

Sol:- Given A = $\begin{bmatrix} 2 & 3 & 7 \\ 3 & -2 & 4 \\ 1 & -3 & -1 \end{bmatrix}$

$R_1 \leftrightarrow R_3$

$\sim \begin{bmatrix} 1 & -3 & -1 \\ 3 & -2 & 4 \\ 2 & 3 & 7 \end{bmatrix}$

$R_2 \rightarrow R_2 - 3R_1$; $R_3 \rightarrow R_3 - 2R_1$

$\sim \begin{bmatrix} 1 & -3 & -1 \\ 0 & 7 & 7 \\ 0 & 9 & 9 \end{bmatrix}$

$C_2 \rightarrow C_2 + 3C_1$; $C_3 \rightarrow C_3 + C_1$

$A \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 7 & 7 \\ 0 & 9 & 9 \end{bmatrix}$

$R_2 \rightarrow R_2 \frac{1}{7}$, $R_3 \rightarrow R_3 . \frac{1}{9}$

$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

$R_3 \rightarrow R_3 - R_2$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$C_3 \rightarrow C_3 - C_2$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} I2 & 0 \\ 0 & 0 \end{bmatrix}$$

Rank of A = ρ (A) = r = 2 = unit matrix order

2) Find the rank of the matrix $\begin{bmatrix} 0 & 1 & 2 & -2 \\ 4 & 0 & 2 & 6 \\ 2 & 1 & 3 & 1 \end{bmatrix}$ by using normal form.

Sol: Given A = $\begin{bmatrix} 0 & 1 & 2 & -2 \\ 4 & 0 & 2 & 6 \\ 2 & 1 & 3 & 1 \end{bmatrix}$

$C_1 \leftrightarrow C_2$

$$A \sim \begin{bmatrix} 1 & 0 & 2 & -2 \\ 0 & 4 & 2 & 6 \\ 1 & 2 & 3 & 1 \end{bmatrix}$$

$R_3 \rightarrow R_3 - R_1$

$$\sim \begin{bmatrix} 1 & 0 & 2 & -2 \\ 0 & 4 & 2 & 6 \\ 0 & 2 & 1 & 3 \end{bmatrix}$$

$C_3 \rightarrow C_3 - 2C,\ C_4 \rightarrow C_4 + 2C_1$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 4 & -6 & 6 \\ 0 & 2 & -3 & 3 \end{bmatrix}$$

$R_3 \rightarrow 2R_3 - R_2$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 4 & -6 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_2 \rightarrow C_2 \cdot \frac{1}{4}$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -6 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_3 \rightarrow C_3 + 6C_2,\ C_4 \rightarrow C_4 - 6C_2$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} I2 & 0 \\ 0 & 0 \end{bmatrix}$$

Rank of A = ρ (A) = r = 2

Inverse of Non-singular matrix by Gauss – Jordan method:-

We can find the inverse of a non-singular square matrix using elementary row operations only.

Suppose A is a nonsingular square matrix of order n we write $A = I_n A$

Now we apply elementary row operations only to the matrix A and the prefactor I_n of the R.H.S. We will do this till we get an equation of the form $I_n = BA$. Then abviously B is the inverse of A.

1) Find the inverse of the Matrix $\begin{bmatrix} 2 & -1 & 3 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$ by using Gaus – Jordan Method

Sol:- Given $A = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$

Write $A = I_n A$

$$\begin{bmatrix} 2 & -1 & 3 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} . A$$

$R_1 \leftrightarrow R_2$

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 3 \\ 1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} . A$$

$R_2 \rightarrow R_2 - 2R_1$; $R_3 \rightarrow R_3 - R_1$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -3 & 1 \\ 0 & -2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & -1 & 1 \end{bmatrix} . A$$

$R_2 \rightarrow R_2 . (\frac{-1}{3})$

$$\begin{bmatrix}1 & 1 & 1\\0 & 1 & -1/3\\0 & -2 & 0\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\-1/3 & 2/3 & 0\\0 & -1 & 1\end{bmatrix}.A$$

$R_1 \to R_1 - R_2$; $R_3 \to R_3 + 2R_2$

$$\begin{bmatrix}1 & 0 & 4/3\\0 & 1 & -1/3\\0 & 0 & -2/3\end{bmatrix} = \begin{bmatrix}1/3 & 1/3 & 0\\-1/3 & 2/3 & 0\\-2/3 & 1/3 & 1\end{bmatrix}$$

$R_3 \to R3(-3/2)$

$$\begin{bmatrix}1 & 0 & 4/3\\0 & 1 & -1/3\\0 & 0 & 1\end{bmatrix} = \begin{bmatrix}1/3 & 1/3 & 0\\-1/3 & 2/3 & 0\\1 & -1/2 & -3/2\end{bmatrix}.A$$

$R_1 \to R_1 - 4/3.R_3$; $R_2 \to R_2 + 1/3.R_3$

$$\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} = \begin{bmatrix}-1 & 1 & 2\\0 & 1/2 & -1/2\\1 & -1/2 & -3/2\end{bmatrix}.A$$

$I_{3x3} = B.A$ where $B = \begin{bmatrix}-1 & 1 & 2\\0 & 1/2 & -1/2\\1 & -1/2 & -3/2\end{bmatrix}$ is the inverse of given matrix.

Exercise:

Find the inverse of the following matrixes by using Gaugs – Jordan method.

1) $\begin{bmatrix}1 & 2 & 3\\2 & 4 & 5\\3 & 5 & 6\end{bmatrix}$

2) $\begin{bmatrix}-2 & 1 & 3\\0 & -1 & 1\\1 & 2 & 0\end{bmatrix}$

3) $\begin{bmatrix}-1 & -3 & 3 & -1\\1 & 1 & -1 & 0\\2 & -5 & 2 & -3\\-1 & 1 & 0 & 1\end{bmatrix}$

Solution of linear System of equations:

An equation of the form $a_1x_1+a_2x_2+a_3x_3+\ldots\ldots+a_nx_n = b$ ……………(1)

Where $x_1,x_2\ldots\ldots, x_n$ are unknowns and $a_1,a_2,\ldots\ldots, a_n$, b are constants is called a linear equations in n unknowns consider the system of m linear equations in n unknowns .

$x_1,x_2\ldots\ldots, x_n$ as given below

$$a_{11}x_1+a_{12}x_2+\ldots\ldots+a_{1n}x_n = b1$$

$$a_{21}x_1 + a_{22}x_2+ \ldots\ldots +a_{2n}x_n = b2$$

……………………………

……………………………

$$a_{m1}x_1 + a_{m2}x_2+ \ldots\ldots +a_{mn}x_n = bm \quad \ldots\ldots\ldots\ldots(2)$$

where aij's and $b_1,b_2 \ldots\ldots\ldots b_m$ are constants. An ordered n- tuple $(x_1,x_2\ldots\ldots, x_n)$ satisfying all equations in (2) is called a solution of the system (2).

The System of equations in (2) can be written in matrix from A X= B ………(3)

Where A = [aij], $x = (x_1,x_2\ldots\ldots, x_n)^T$, $B = (b_1,b_2\ldots\ldots b_m)^T$

The Matrix [A/B] is called the augmented matrix of the system(2)

If B=0 in (3), the system is said to be Homogeneous otherwise the system is said to be non – homogeneous.

* The system AX = 0 is always consistent since X = 0 (i.e., $x_1=0$, $x_2=0$, …. $X_n=0$) is always a solution of AX = 0 This solution is called Trival solution of the system.

* Given AX = 0, we try to decide whether it has a solution $X \neq 0$. Such a solution, if exists, is called a non-Trival solution

* If there is a least one solution for the given system is said to consistent, if the system does not have any solution, the system is said to be inconsistent.

Solution of Non-homogeneous system of equations:

The system AX=B is consistent i.e., it has a solution (unique or infinite) if and only if rank A = rank [A/B]

i) If rank of A = rank of [A/B] = r<n then the system is consistent and it has infinitely many solutions. There r = rank, n = number of unknowns in the system.

ii) If rant of A = rank of [A/B] = r = n then the system has unique solution.

iii) If rank of A≠ rank [A/B] then the system is inconsistent i.e., It has no solution.

Solved Problems:

1) Solve the system of equations x+2y+3z=1; 2x+37+8z=2 ; x+y+z = 3

Sol: Given system can be written in matrix form

as $\begin{bmatrix}1 & 2 & 3\\2 & 3 & 8\\1 & 1 & 1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}1\\2\\3\end{bmatrix}$

A X = B

Augmented matrix of the given system

$[A/B] = \begin{bmatrix}1 & 2 & 3 & 1\\2 & 3 & 8 & 2\\1 & 1 & 1 & 3\end{bmatrix}$

$R_2 \rightarrow R_2 - 2R_1$; $R_3 \rightarrow R_3 - R_1$

$\sim \begin{bmatrix}1 & 2 & 3 & 1\\0 & -1 & 2 & 0\\0 & -1 & -2 & 2\end{bmatrix}$

$R_3 \rightarrow R_3 - R_2$

$\sim \begin{bmatrix}1 & 2 & 3 & 1\\0 & -1 & 2 & 0\\0 & 0 & -4 & 2\end{bmatrix}$

∴ rank of A = rank [A/B] = r = 3 = number of unknowns = n

∴ n = r = 3

∴ The given system is consistent and it has unique solution. The solution is as follows from the last augmented matrix we can write as

$-4z = 2$	$-y+2z = 0$	$x+2y+3z = 1$
$z = \frac{-1}{2}$	$2z = y$	$x = 1-2y-3z$
	$2(\frac{-1}{2}) = y$	$= 1-2(-1)-3(\frac{-1}{2})$
	$Y=-1$	$= 1+2+\frac{3}{2}$
		$X = 9/2$

∴ The solution of given system : x=9/2; y= -1, z=-1/2

2) Solve the system of equations $x+2y+z=14$

$$3x+4y+z=11$$

$$2x+3y+z=11$$

Sol:- Given system can be written in matrix form as

$$\begin{bmatrix} 1 & 2 & 1 \\ 3 & 4 & 1 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 14 \\ 11 \\ 11 \end{bmatrix}$$

$$A \quad X \quad = B$$

The augmented matrix of the given system as

$$[A/B] = \begin{bmatrix} 1 & 2 & 1 & 14 \\ 3 & 4 & 1 & 11 \\ 2 & 3 & 1 & 11 \end{bmatrix}$$

$R_2 \to R_2 - 3R_1$; $R_3 \to R_3 - 2R_1$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 14 \\ 0 & -2 & -2 & -31 \\ 0 & -1 & -1 & -17 \end{bmatrix}$$

$R_3 \to 2R_3 - R_2$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 14 \\ 0 & -2 & -2 & -31 \\ 0 & 0 & -0 & -3 \end{bmatrix}$$

Rank of A = 2≠3 = rank of AB

∴ The given system has no solution, i.e., the system is inconsistent

3) Show that the system $x+y+z=6$; $x+2y+3z = 14$; $x+4y+7z = 30$ are consistent and solve them.

Sol:- Given system can be written in matrix form as

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 14 \\ 30 \end{bmatrix}$$

Augmented matrix

$$[A/B] = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 14 \\ 1 & 4 & 7 & 30 \end{bmatrix}$$

$R_2 \to R_2 - R_1$; $R_3 \to R_3 - R_1$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 8 \\ 0 & 3 & 6 & 24 \end{bmatrix}$$

$R_3 \to R_3 - 3R_2$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Rank of A = rank of AB = r = 2 < 3 = n = number of unknowns

∴ The system has consistent and it has infinitely many solutions.

Here $x+y+z = 6$

$Y+2z = 8$

Let $z = k$

Now $y = 8-2z=8-2k$

Now $x = 6-y-z$

$= 6-(8-2k)-k$

$x= 6-8+2k-k$

$x= k-2$

∴ The system has infinitely many solutions $x=k-2$; $y=8-2k$; $z=k$

p) For what values of λ and μ the system of equations

2x+3y+5z = 9 have (i) no solution

7x+3y-2x = 8 (ii) unique solution

2x+3y+lz = μ (iii) infinitely many solutions

The matrix form of given system of equations

$$\begin{bmatrix} 2 & 3 & 5 \\ 7 & 3 & -2 \\ 2 & 3 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 8 \\ \mu \end{bmatrix}$$

The augumented matrix of given system

$$[A/B] = \begin{bmatrix} 2 & 3 & 5 & 9 \\ 7 & 3 & -2 & 8 \\ 2 & 3 & \lambda & \lambda \end{bmatrix}$$

$R_2 \rightarrow 2R_2 - 7R1$; $R_3 \rightarrow R_3 - R_1$

$$\sim \begin{bmatrix} 2 & 3 & 5 & 9 \\ 0 & -15 & -39 & -47 \\ 0 & 0 & \lambda - 5 & \mu - 9 \end{bmatrix}$$

$R_1 \rightarrow R1(\frac{1}{2})$

$$\sim \begin{bmatrix} 1 & 3/2 & 5/2 & 9/2 \\ 0 & -15 & -39 & -47 \\ 0 & 0 & \lambda - 5 & \mu - 9 \end{bmatrix}$$

Case 1 : λ=5, μ≠9

Then ρ (A) = 2, ρ (AB) = 3

ρ (A) = 2≠3= ρ (AB)

The system has no solution

Case 2:- λ≠5, μ≠9

Then ρ (A) = ρ (A/B) = r=n=3

∴ The system has unique solution

Case 3: λ=5, μ=9

Then ρ (A) = ρ (A/B)=r=2<3=n=number of unknowns

∴ The system has infinitely many solutions

Consistency of system of homogeneous linear equations:

Consider of system of homogeneous linear equations in n unknowns namely

$$a_{11}x_1 + a_{12}x_2 + \ldots\ldots + a_{1n}x_n = 0$$

$$a_{21}x_1 + a_{22}x_2 + \ldots\ldots + a_{2n}x_n = 0$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$a_{m1}x_1 + a_{m2}x_2 + \ldots\ldots + a_{mn}x_n = 0$$

This system can be written in matrix form

$$\begin{bmatrix} a_{11} & a_{12} & \cdots\cdots & a_{1n} \\ a_{21} & a_{22} & \cdots\cdots & a_{2n} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ a_{m1} & a_{m1} & \cdots\cdot & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ 0 \end{bmatrix}$$

$$A \quad X = 0$$

1. If rank of A = n (number of variables)

⇒ The system of equations have only trivial solution (i.e., zero solution)

2. If r<n then the system have an infinitive number of non trivial solutions.

Solved Problems:

1) Find all the solutions of the system of equations

X+2y-z=0, 2x+y+z=0, x-4y+5z=0

Sol. Given system can be written in matrix form

$$\begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 1 \\ 1 & -4 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Augmented matrix

$$[A/B] = \begin{bmatrix} 1 & 2 & -1 & 0 \\ 2 & 1 & 1 & 0 \\ 1 & -4 & 5 & 0 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$; $R_3 \to R_3 - R_1$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & -6 & 6 & 0 \end{bmatrix}$$

$R_3 \to R_3 - 2R_2$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Rank of A = rank of AB = r = number of non zero rows = 2<3=n= number of variables

$\therefore$ The system has infinitely many solutions from the above matrix

-3y+3z=0 x+2y-z=0

$\Rightarrow$ y=z

Let us consider n-r=3-2=1 arbitrary constants

Let z=k, then y = k

Since x+2y-z=0

$\Rightarrow$ x=z-2y

= z-2y

=k-2k

x=-k

$\therefore$ x=-k, y=z=k

2) Solve the system of equations x+y+w=0; y+z=0, x+y+z+w=0, x+y+2z=0

Sol: Given system can be written in matrix form

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Augmented matrix

$$[A/B] = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 \end{bmatrix}$$

$R_3 \to R_3 - R_1$; $R_4 \to R_4 - R_1$

$$\sim \begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & -1 & 0 \end{bmatrix}$$

$R_4 \to R_4 - 2R_3$

$$\sim \begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \end{bmatrix}$$

$R_1 \to R_1 + R_4$

$$\sim \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \end{bmatrix}$$

Rank of A = Rank of AB = r =4=n= number of unknowns

∴ Therefore there is no non=zero solution

∴ x=y = z=w=0 is only the trivial solution.

Matrix: A set of mn real or complex numbers or functions displayed as an array of m horizontal lines (called rows) and n vertical lines (called columns) is called a matrix of order (m, n) or $m \times n$ (read as m by n) .The numbers or functions are called the elements or entries of the matrix and are enclosed within brackets [ ] or ()

Matrices are denoted with capital letters A , B , C ..& elements are denoted with small letters a ,b , c ….letters i and j are used as suffixes on the a , b , c ….to denote the row and columns position respectively of the corresponding entry .

Thus

$$A=[a_{ij}]=\begin{bmatrix} a_{11} & a_{12\ldots} & a_{1j\ldots} & a_{1n} \\ a_{21} & a_{22\ldots} & a_{2j\ldots} & a_{2n} \\ a_{i1} & a_{i2\ldots} & a_{ij\ldots} & a_{in} \\ a_{m1} & a_{m2\ldots} & a_{mj\ldots} & a_{mn} \end{bmatrix} \quad \text{where } 1\le i\le m$$

$$1\le j \le n$$

is a matrix with m rows and n columns

Types of matrices :

Real matrix: A matrix whose elements are all real numbers or function is called a real matrix

Ex : $\begin{bmatrix} -1 & 0 \\ 2 & -2 \\ 13 & 5 \end{bmatrix}$, $\begin{bmatrix} e^x & y \\ 0 & -1 \end{bmatrix}$.

Complex matrix: A matrix which contains at least one complex numbers or function as on element is called a complex matrix

Ex : $\begin{bmatrix} 1 & -i \\ 0 & 2 \end{bmatrix}$, $\begin{bmatrix} 7 & 3+i \\ 13 & 8 \end{bmatrix}$

Row matrix: A matrix with only one row is called a row matrix or row vector .It is a matrix of order $1 \times n$ for some positive integer n.

Ex :[-3 7 0 2 11] ; [7 4 8]

Column matrix: A matrix with only one column is called a column matrix or column vector .It is a matrix of order $m \times 1$ for some positive integer m .

Ex: $\begin{bmatrix} 0 \\ 2 \\ 16 \end{bmatrix}$ & $\begin{bmatrix} 5 \\ 12 \\ 6 \end{bmatrix}$

Square matrix: A matrix in which the number of rows and the number of columns are equal is called a square matrix

Ex: $\begin{bmatrix} 1 & -2 \\ 0 & 5 \end{bmatrix}$, $\begin{bmatrix} 0 & 5 & 3 \\ 7 & 6 & 4 \\ -3 & 0 & 2 \end{bmatrix}$

A square matrix of order $n \times n$ is simply described as an n- square matrix.
Diagonal matrix: A square matrix [aij] with $aij = 0$ for $i \neq j$ is called a diagonal matrix

Ex: $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$ & $\begin{bmatrix} 5 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & -8 \end{bmatrix}$; $\begin{bmatrix} 11 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Scalar matrix: A diagonal matrix which consists all the elements are equal in the diagonal is called scalar matrix.

Ex: $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$, $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

Zero or null matrix : A matrix in which every entry is zero is called a zero matrix or null matrix and is denoted by o.

EX: $0_{3\times2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$, $0_{1\times2} = [0\ 0]$

Unit matrix (or) Identity matrix :A diagonal matrix in which all the diagonal elements are equal to unity or 1 is called unit matrix (or) Identity matrix and is denoted by I.

Ex: $I = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}_{3\times3}$; $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2\times2}$

Rectangular matrix: A matrix in which the numbers of rows and the numbers of columns may not be equal is called a rectangular matrix .

Transpose of a matrix: The matrix obtained from any given matrix A, by interchanging its rows and cplumns is called the transpose of A and it is denoted by A^1or A^T

Ex: $A=\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

$A^T=\begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$

Properties of transpose of a matrix:

If A^T and B^T be the transposes of A and B respectively , then

1) $(A^T)^T = A$
2) $(A+B)^T = A^T+B^T$, A and B being of the same order
3) $(KA)^T = K.A^T$, K is a scalar
4) $(AB)^T = B^T A^T$, A and B being conformable for multiplication

Trace of a square matrix : The sum of the elements along the main diagonal of a square matrix ,A is called the trace of A and written as

Trace(A)= $a_{11}+a_{22}+.....a_{nn}$

$Tr(A)=\sum_{i=1}^{n} a_{ij}$

Properties of trace of A

$Tr(KA) = K. Tr(A)$, where K is a scalar

$Tr(A+B) = Tr(A) + Tr(B)$

$Tr(AB) = Tr(BA)$

Minor and cofactor:

Let $A=[aij]_{n\times n}$ be a square matrix when from A the elements of i^{th} row and j^{th}

column are deleted the determinant of (n-1) rowed matrix M_{ij} is

called the minor of a_{ij} of A and is denoted by $|M_{ij}|$. the signed minor $(-1)^{i+j}$, $|M_{ij}|$ is

called the cofactor of a_{ij} and is denoted by A_{ij}

Ex:

let $A=\begin{bmatrix}1 & 2 & 3\\ 4 & 5 & 6\\ 7 & 8 & 9\end{bmatrix}_{3\times 3}$

Minor of 1 is $=|(5\times 9)-(6\times 8)|$

$=|45-48|$

$=|-3|=3$

Cofactor of 1 is (-1)

Adjoint of a square matrix: let A be a square matrix of order n .The transpose of the matrix got from A by replacing the elements of A by the corresponding cofactors is called adjoint of A and is denoted by ad j A .

Singular & non singular matrices:

A square matrix 'A' is said to be singular if $|A|=0$

If $|A|\neq 0$ then A is said to be non-singular .

Invertible matrix : A square matrix A is said to be invertible if there exists a matrix B such that AB=BA=I is called an inverse of A.

Note:

1) A matrix is said to be invertible ,if it posses inverse

2)Every invertible matrix possesses a unique inverse

(or)

The inverse of a matrix if it exists is unique.

3) The inverse of A is denote by A^{-1} thus $AA^{-1}=A^{-1}A=I$

4) If A is an invertible matrix and if A=B then $A^{-1}=B^{-1}$

5) If $|A| \neq 0$ then $A^{-1}=\frac{1}{|A|}$.(adj A).

Symmetric matrix : A square matrix $A=[a_{ij}]$ is said to be symmetric if $a_{ij}=a_{ji}$ for every ; iand j thus A is symmetric matrix if $A=A^T$ (or) $A^T=A$

Ex: $A=\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 7 \end{bmatrix}$

$A^T=\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 7 \end{bmatrix}$

$\therefore A=A^T$ hence A is symmetric.

Skew –symmetric : A square matrix $A=[a_{ij}]$ is raid to be skew-symmetric if $a_{ij}=-a_{ji}$ for every i and j
Thus A is skew symmetric $\Leftrightarrow A=-A^T$

Ex : Let $A=\begin{bmatrix} 0 & a & -b \\ -a & 0 & c \\ b & -c & 0 \end{bmatrix}$

$$A^T=\begin{bmatrix} 0 & -a & b \\ a & 0 & -c \\ -b & c & 0 \end{bmatrix}=-\begin{bmatrix} 0 & a & -b \\ -a & 0 & c \\ b & -c & 0 \end{bmatrix}=-A$$

$\therefore A^T=-A \Rightarrow A=-A^T$

$\therefore$ A is skew –symmetric

Orthogonal matrix: A square matrix A is raid to be orthogonal if $AA^T=A^TA=I$.
That is $A^T=A^{-1}$

Conjugate of a matrix: The matrix obtained form any given matrix A, on replacing its element by the corresponding conjugate complex numbers is called the conjugate of A and is denoted by $\bar{A}$

Ex :$A=\begin{bmatrix} 2 & 3i & 2-5i \\ -i & 0 & 4i+3 \end{bmatrix}$, then $\bar{A}=\begin{bmatrix} 2 & -3i & 2+5i \\ i & 0 & -4i+3 \end{bmatrix}$

The transport of the conjugate of a square matrix

If A is a square matrix and its conjugate is $\bar{A}$, then the transpose of $\bar{A}$ is $(\bar{A})^T$. It can be easily seen that $(\bar{A})^T=\overline{(A^T)}$. The transposed conjugate of A is denoted by A^θ

$A^\theta=(\bar{A})^T=\overline{(A^T)}$.

Note :

1) $(A^\theta)^\theta= A$
2) $(A\pm B)^\theta = A^\theta \pm B^\theta$
3) $(KA)^\theta = \bar{K} A^\theta$ where k is a complex number
4) $(AB)^\theta = A^\theta B^\theta$

Hermitian matrix: A square matrix A such that $A^T= \bar{A}$ or $(\bar{A})^T=A$ is called a Hermitian matrix .

Ex : $A=\begin{bmatrix} 4 & 1+3i \\ 1-3i & 7 \end{bmatrix}$

$\bar{A}=\begin{bmatrix} 4 & 1-3i \\ 1+3i & 7 \end{bmatrix}$ & $A^T=\begin{bmatrix} 4 & 1-3i \\ 1+3i & 7 \end{bmatrix}$

$\bar{A}=A^T$

∴ A is hermitian

Skew –Her mitian matrix:

A square matrix A such that $A^T = -\bar{A}$ or $(\bar{A})^T$=-A is called a skew-Hermitian matrix

Ex : A=$\begin{bmatrix} -3i & 2+i \\ 2+i & -i \end{bmatrix}$

$\bar{A}$=$\begin{bmatrix} 3i & 2-i \\ -2-i & i \end{bmatrix}$, A^T= $\begin{bmatrix} -3i & -2+i \\ 2+i & -i \end{bmatrix}$

$(\bar{A})^T$=$\begin{bmatrix} 3i & -2-i \\ 2-i & i \end{bmatrix}$

-A= $\begin{bmatrix} 3i & -2-i \\ 2-i & i \end{bmatrix}$

∴ $(\bar{A})^T$= -A

∴ A is skew –Hermition

Unitary matrix : A square matrix A is said to be unitary if $A^{\theta}.A = A.A^{\theta} = I$.

Gauss elimination method:-

This method of solving a system of n liner equations in n unknowns consists of eliminating the coefficients in such a way that the system reduces to upper triangular system which may be solved by back substitution.

Problems:

Solve the equations x+y+z=6, 3x+3y+4z=20, 2x+y+3z=13 by using Gauss elimination method.

Sol matrix from of the given system

$$\begin{bmatrix} 1 & 1 & 1 \\ 3 & 3 & 4 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 20 \\ 13 \end{bmatrix}$$

Augmented matrix of the given system

$$[A/B] = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 3 & 3 & 4 & 20 \\ 2 & 1 & 3 & 13 \end{bmatrix}$$

$R_2 \rightarrow R_2 - 3R_1$; $R_3 \rightarrow R_3 - 2R_1$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 0 & 1 & 2 \\ 0 & -1 & 1 & 1 \end{bmatrix}$$

$R_2 \rightarrow R_3$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

Clearly it is an upper triangular matrix from this by back substitution.

z= 2	-y+z=1	x+y+z=6
	z-1=y	x=6-y-z
	2-1=y	=6-1-2
	Y=1	=3
∴ x=3	y=1	z=2

Gauss Seidel iteration method:

We will consider the system of equations

$$a_{11}x_1+a_{12}x_2+a_{13}x_3 = b_1 \quad \ldots\ldots(1)$$

$$a_{21}x_1 + a_{22}x_2+ a_{23}x_3 = b_2 \quad \ldots\ldots(2)$$

$$a_{31}x_1 + a_{32}x_2+ a_{33}x_3 = b_3 \quad \ldots\ldots(3)$$

Where the diagonal coefficients are not zero and are large compared to other coefficients such a system is called a "diagonally dominant system".

The system of equations (1) can be written as

$$x_1=\frac{1}{a_{11}}[b_1-a_{12}x_2-a_{13}x_3]\ldots\ldots\ldots\ldots(4)$$

$$x_2=\frac{1}{a_{22}}[b_2-a_{21}x_1-a_{23}x_3]\ldots\ldots\ldots\ (5)$$

$$x_3=\frac{1}{a_{33}}[b_3-a_{31}x_1-a_{32}x_2]\ldots\ldots\ldots\ldots(6)$$

Let the initial approximate solution be $x_1^{(0)}, x_2^{(0)}, x_3^{(0)}$ are zero Substitute x_2^0, x_3^0 in (4) we get

$x_1^1 = 1/a_{11}\,[b_1-a_{12}x_2^0-a_{13}x_3^0]$ this is taken as first approximation of x_1

Substitute x_1^1, x_3^0 in (5) we get $x_2^1 = 1/a_{22}[b_2-a_{21}x_1^1-a_{23}x_3^0]$

This is taken as first approximation of x_2 now substitute x_1^1, x_2^1 in (6), we get

$x_3^1 = 1/a_{33}\, b_3-a_{31}x_1^1-a_{32}x_2^1]$

This is taken as first approximation of x_3 continue the same procedure until the desired order of approximation is reached or two successive iterations are nearly same. The final values of x_1, x_2, x_3 obtained an approximate solution of the given system.

1) Use Gauss-Seidel iteration method to solve

$10x+y+z=12;\ 2x+10y+z=13;\ 2x+2y+10z=14$

Sol: Clearly the given system is diagonal by dominant and we write it as

$x = \frac{1}{10}(12-y-z)$ (1)

$y = \frac{1}{10}(13-2x-z)$ (2)

$z = \frac{1}{10}(14-2x-2y)$ (3)

First iteration: We start iteration by taking $y=z=0$ in (1) we get $x_1^1=1.2$

Put $x^1 = 1.2,\ z = 0$ in (2) we get $y^1 = 1.06$

Put $x^1 = 1.2;\ y^1 = 1.06$ (3) we get $z^1 = 0.95$

Second iteration now substitute $y^1 = 1.06,\ z^1 = 0.95$ in (1)

$x^2 = \frac{1}{10}(12-1.06-0.95) = 0.999$

put $x^2,\ z^1$ in (2) $y^2 = \frac{1}{10}(13-1.998-0.95) = 1.005$

now substitute $x^2,\ y^2$ in (3) $z^2 = \frac{1}{10}(14-1.998-2.010) = 0.999$

Third approximation: now substitute $y^2,\ z^2$ in (1)

$x^3 = \frac{1}{10}(12-1.005-0.999) = 1.00$

Put x^3, z^3 in (2) $y^3 = \frac{1}{10}(13-2.0-0.999) = 1.000$

Put y^3, x^3 in (3) $x^3 = \frac{1}{10}(14-2.0-2.0) = 1.00$

Similarly we find fourth approximation of x,y,z and got them as $x^4=1.00,\ y^4=1.00,\ z^4=1.00$

Exercise:

Solve the following system of equations by Gauss – seided method

1) $8x-3y+2z = 20;\ 4x+11y-z=33,\ 6x+3y+12z = 36$

2) $x+10y+z = 6;\ 10x+y+z=6;\ x+y+10z=6$

UNIT - II

Eigen Values and Eigen Vectors

Eigen Values:-

Let $A = [a_{ij}]_{n \times n}$ be a square matrix of order n & λ is the scalar quantity, is called the

1) The Matrix $A - \lambda I$ is called the characteristic Matrix is A where I is the unit matrix of order n.

2) The polynomial $|A - \lambda I|$ in λ of degree n is called characteristic polynomial of A.

3) The equation $|A - \lambda I| = 0$

i.e., $\begin{bmatrix} a11-\lambda & a12 & . & a1n \\ a21 & a22-\lambda & . & a2n \\ . & . & . & . \\ an1 & an2 & . & ann-\lambda \end{bmatrix} = 0$ is called characteristic equation of A

Note:- The characteristic equation is of the form $(-1)^n\lambda^n + C_1\lambda^{n-1}+c_2\lambda^{n-2}+\ldots+c_n=0$

4) The roots of the characteristic equation $|A - \lambda I| = 0$ are called characteristic roots (or) latent roots (or) Eigen values of the Matrix A.

Note: 1. The set of all eigen values of A is called the Spectrum of A.

2. The degree of the characteristic polynomial is equal to the order of the matrix.

Eigen Vectors:-

Let $A = [a_{ij}]_{n \times n}$, A non – zero vector x is said to be a characteristic vector of A if λ a scalar λ such that $AX = \lambda X$.

If $AX = \lambda X$, $(x \neq 0)$ we say that x is Eigen vector or characteristic vector of A corresponding to the Eigen value or characteristic value λ of A.

Solved Problems:

1) Find the Eigen values of $A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$

Sol:- Step 1:- Given Matrix $A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$

Step 2:- Characteristic equation $|A - \lambda I| = 0$

$= \begin{bmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{bmatrix} = 0$

$(5-\lambda)(2-\lambda) - 4 = 0$

$10-5\lambda-2\lambda+\lambda^2-4=0$

$\lambda^2-7\lambda+6=0$

Step 3:- The roots of characteristic equation is called eigen values or eigen roots or latent values.

$\lambda^2-7\lambda+6=0$

$\lambda^2-6\lambda-\lambda+6=0$

$\lambda(\lambda-6) -1(\lambda-6) = 0$

$(\lambda-6)(\lambda-1) = 0$

$\lambda = 1, 6$

$\therefore$ Eigen values are 1,6

2) Find the characteristic roots or eigen roots of $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & -1 \\ 2 & -1 & 0 \end{bmatrix}$

Sol:- Step1: Given matrix $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & -1 \\ 2 & -1 & 0 \end{bmatrix}$

Step 2: Characteristic Equation

$|A-\lambda I| = 0$

$$\begin{bmatrix} 0-\lambda & 1 & 2 \\ 1 & 0-\lambda & -1 \\ 2 & -1 & 0-\lambda \end{bmatrix} = 0$$

$\Rightarrow -\lambda^3+6\lambda-4 = 0$

$\lambda^3-6\lambda+4 = 0$

Step 3: roots of above egn are called eigen values.

$\lambda^3-6\lambda-4 = 0$

$(\lambda-2)(\lambda^2+2\lambda-2) = 0$

$\lambda = 2, \lambda = \frac{-2\pm\sqrt{4+8}}{2}$

$\lambda = 2, -1\pm\sqrt{3}$

Eigen roots are 2, $-1\pm\sqrt{3}$

Exercise problems:-

1) Find the eigen values $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$

2) Find the eigen values $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & 2 \end{bmatrix}$

3) Find the eigen values $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$

4) Find the eigen values $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

Eigen vector problems

1) Find the Eigen values and Eigen vectors of the following matrix $A = \begin{bmatrix} 5 & -2 & 0 \\ -2 & 6 & 2 \\ 0 & 2 & 7 \end{bmatrix}$

Sol: Step1:- given matrix $A = \begin{bmatrix} 5 & -2 & 0 \\ -2 & 6 & 2 \\ 0 & 2 & 7 \end{bmatrix}$

Step2:- Characteristic equation $|A-\lambda I| = 0$

$$\begin{bmatrix} 5-\lambda & -2 & 0 \\ -2 & 6-\lambda & 2 \\ 0 & 2 & 7-\lambda \end{bmatrix} = 0$$

$(5-\lambda)\{(6-x)(7-\lambda)-4\} +2\{-2(7-\lambda) -0\} +0 = 0$

$-\lambda^3+18\lambda^2-99\lambda+162=0$

$\lambda^3-18\lambda^2+99\lambda-162=0$

Step3:-$(\lambda-3)(\lambda-6)(\lambda-9) = 0$

$\lambda=3, 6,9$

$\therefore$ Eigen values are 3,6,9

Step3: Eigen vectors

1) Eigen vector corresponding to $\lambda= 3$ $[A-\lambda I]x = 0$; $[A-3I]x = 0$

$$\begin{bmatrix} 5-3 & -2 & 0 \\ -2 & 6-3 & 2 \\ 0 & 2 & 7-3 \end{bmatrix}\begin{bmatrix} x1 \\ x2 \\ x3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -2 & 0 \\ -2 & 3 & 2 \\ 0 & 2 & 4 \end{bmatrix}\begin{bmatrix} x1 \\ x2 \\ x3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Using Echelon form

$$R_2\to R_2+R_1\begin{bmatrix} 2 & -2 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \end{bmatrix}\begin{bmatrix} x1 \\ x2 \\ x3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_3\to R_3+2R_1\begin{bmatrix} 2 & -2 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x1 \\ x2 \\ x3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Rank = 2 = no. of non zero rows

N = no. of unknowns (or) no. of variables n = 3

$r<n \Rightarrow n-r = 3-2 = 1$ we choose one variable to the one constant.

$2x_1-2x_2 = 0$

$x_1+2x_3 = 0$

let $x_3 = k$

$2x_1 = 2x_2 = 2[-2k] = -4k$

$x1 = \frac{-4}{2}k = - 2k$

$$\text{Eigen vector } x_1 = \begin{bmatrix} x1 \\ x2 \\ x3 \end{bmatrix} = \begin{bmatrix} -2k \\ -2k \\ k \end{bmatrix} = k\begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix}$$

Eigenvector corresponding to 6 :- $[A-6I]x = 0$

Using Echelon form

$$R_2\to R_2-2R_1\begin{bmatrix} -1 & -2 & 0 \\ 0 & 4 & 2 \\ 0 & 2 & 1 \end{bmatrix}\begin{bmatrix} x1 \\ x2 \\ x3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_3\to 2R_3-R_2\begin{bmatrix} -1 & -2 & 0 \\ 0 & 4 & 2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x1 \\ x2 \\ x3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$r = 2$, $n = 3$

we choose one variable to the one constant.

i.e., $x3=k$

$-x_1-2x_2 = 0$

$4x_2+2x_3 = 0$

$x_3 = k$

$4x_2= -2x_3 = -2k$

$X_2 = -\frac{1}{2} = k$

$-x_1-2x_2 = 0 \Rightarrow -x_1 = 2x_2 = 2[\frac{-1}{2}]\, k$

$x_1 = k,\ x_2 = \frac{-1}{2}k,\ x_3 = k,$

Eigen vector $x_2 = \begin{bmatrix} x1 \\ x2 \\ x3 \end{bmatrix} = \begin{bmatrix} k \\ -1/2k \\ k \end{bmatrix}$

$x_2 = \frac{k}{2}\begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$

Eigenvector corresponding to 9 :- [A-9I]x = 0

$$\begin{bmatrix} -4 & -2 & 0 \\ -2 & -3 & 2 \\ 0 & 2 & -2 \end{bmatrix}\begin{bmatrix} x1 \\ x2 \\ x3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \rightarrow 2R_2-R_1 \begin{bmatrix} -4 & -2 & 0 \\ 0 & -4 & 4 \\ 0 & 2 & -2 \end{bmatrix}\begin{bmatrix} x1 \\ x2 \\ x3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$R_3 \rightarrow 2R_3-R_2 \begin{bmatrix} -4 & -2 & 0 \\ 0 & -4 & 4 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x1 \\ x2 \\ x3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$r=2,\ n = 3$

$n-r= 3-2 = 1$

Let $x_3 = k$

$-4x_1 - 2x_2 = 0$

$-4x_2 + 4x_3 = 0$

$-x_2 = -x_3$

$x_2 = x_3 = k$

$-4x_1-2x_2 = 0$

$-2x_1 = x_2$

$x_2 = -2x_1 = -2k$

$x_1 = \frac{-x2}{2} = \frac{-k}{2}$

$\therefore$ Eigen vector $x_3 = \begin{bmatrix} x1 \\ x2 \\ x3 \end{bmatrix} = \begin{bmatrix} -k/2 \\ k \\ k \end{bmatrix}$

$$x_3 = \frac{k}{2}\begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$$

Three eigen vectors are

$$x_1 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}, \qquad x_2 = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}, \qquad x_3 = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$$

2) Find the characteristic roots and find the corresponding eigen vectors $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

Sol :- Step1: Given Matrix A = $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

Step 2:- Characteristic Egn $|A-\lambda I| = 0$

$$\begin{bmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{bmatrix} = 0$$

$\Rightarrow \lambda^3-12\lambda^2+36\lambda-32 = 0$

$\Rightarrow (\lambda-2)(\lambda^2-10\lambda+16) = 0$

$(\lambda-2)(\lambda-2)(\lambda-8) = 0$

$\lambda = 2, 2, 8$

Step 3:- Eigen values are 2,2,8

Eigen Vectors:- The eigen vector of A Corresponding to $\lambda = 2$

$[A-\lambda I]x = 0$, $[A-2I]x = 0$

$$\begin{bmatrix} -4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix}\begin{bmatrix} x1 \\ x2 \\ x3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The eigen vector of A corresponding to $\lambda = 8$

$[A-8I]x = 0$

$$\begin{bmatrix} -2 & -2 & 2 \\ -2 & -5 & -1 \\ 2 & -1 & -5 \end{bmatrix}\begin{bmatrix} x1 \\ x2 \\ x3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \rightarrow R_2-R_1$; $R_3 \rightarrow R_3-R_1$ $\begin{bmatrix} -2 & -2 & 2 \\ 0 & -3 & -3 \\ 2 & -3 & -3 \end{bmatrix}\begin{bmatrix} x1 \\ x2 \\ x3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$R_3 \rightarrow R_3-R_1$ $\begin{bmatrix} -2 & -2 & 2 \\ 0 & -3 & -3 \\ 2 & 0 & 0 \end{bmatrix}\begin{bmatrix} x1 \\ x2 \\ x3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

r=2, n = 3, 1-r = 3-2 = 1 we have to select one variable to the one constant i.e, $x_3 = k$

$-2x_1-2x_2+2x_3 = 0$

$-3x_2+(-3)x_3 = 0$

$x_2 = -x_3 = -k$

$x_1 = 2k$

$$\Rightarrow x_3 = \begin{bmatrix} x1 \\ x2 \\ x3 \end{bmatrix} = \begin{bmatrix} 2k \\ -k \\ k \end{bmatrix} = k \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

$$\therefore \text{ Eigen vectors are } x_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, x_2 = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}, x_3 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

Exercise problems

I. Find the eigen values & Eigen vectors of the following matrixs.

1) $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

Ans:- $\lambda = 0,0,3$ Eigen Vectors $\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

2) $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

Ans:- $\lambda = 0,3,15$ Eigen Vectors $\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$

3) $A = \begin{bmatrix} 3 & 1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$

Ans:- $\lambda = 2,3,6$ Eigen Vectors $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

4) $A = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

Ans:- $\lambda = 1,3,6$ Eigen Vectors $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$

5) $A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 2 & 2 \\ 0 & 0 & -2 \end{bmatrix}$

Ans:- $\lambda = 1,2,-2$ Eigen Vectors $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -4/3 \\ 1 \\ -2 \end{bmatrix}$

6) $A = \begin{bmatrix} 1 & 3 & 4 \\ 0 & 2 & 5 \\ 0 & 0 & 3 \end{bmatrix}$

Ans:- $\lambda = 1,2,3$ Eigen Vectors $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 19 \\ 10 \\ 2 \end{bmatrix}$

Eigen values of Hermitian, Skew Hermitian and Unitary Matrix

Note:- Hermitian $\Rightarrow A^{\theta} = A$

Skew Hermitian $\Rightarrow A^{\theta} = -A$

Unitary $\Rightarrow AA^{\theta} = I$

Where $A^{\theta} = (\bar{A})^T$

1) Find the eigen values of the following matrix A = $\begin{bmatrix} 4 & 1-3i \\ 1+3i & 7 \end{bmatrix}$ & S.T. Hermitian.

Sol:- A = $\begin{bmatrix} 4 & 1-3i \\ 1+3i & 7 \end{bmatrix}$ Characteristic equation of A is $|A-\lambda I| = 0$

$\begin{bmatrix} 4-\lambda & 1-3i \\ 1+3i & 7-\lambda \end{bmatrix} = 0$

A is Hermitian $A^{\Theta}=A$; Eigen values of Hermitian matrix are real.

Exercise Problems:-

1) S.T. A = $\begin{bmatrix} 3i & 2-i \\ -2+i & i \end{bmatrix}$

Skew Hermitian & find eigen values Ans:- $\lambda = 4i, -2i$

2) S.T. C = $\begin{bmatrix} 1/2i & \sqrt{3}/2 \\ \sqrt{3}/2 & i/2 \end{bmatrix}$

S.T. Unitary & find eigen values Ans:- $-\sqrt{3/2} + i/2$

3) P.T. $1/\sqrt{3} = \begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$

is unitary and determine the Eigen values & Eigen Vectors.

4) S.T. $\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$

is skew hermitian and find the eigen values & eigen vectors.

5) Verify that the matrix A = $1/2 \begin{bmatrix} 1+i & 1-i \\ 1-i & 1+i \end{bmatrix}$ has eigen values with unit modules.

6) Show that A = $\begin{bmatrix} i & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{bmatrix}$ is skew Hermitian and unitary and find the eigen values and eigen vectors.

Diagonalization of a matrix

A matrix A is diagonalizable if there exists an invertible matrix P such that $P^{-1}AP = D$ where D is a diagonal matrix. Also the matrix P is then said to diagonalize A or transform A to diagonal form.

Similarity of Matrix:- Let A & B be square matrices to A It ∃ a non – singular matrix P of order n → B $P^{-1}AP$. It is denoted by A B. The transformation y = Px is called similarity transformation.

Thus a matrix is said to be diagonalizable if it is similar to a diagonal matrix.

Note:- A is nxn matrix. Then A is similar to a diagonal matrix D = diag $[\lambda_1, \lambda_2, \ldots.. \lambda_n]$

$\therefore$ An invertible matrix $P = [x_1, x_2 \ldots.. x_n] \rightarrow P^{-1}AP = D = \text{diag}\,([\lambda+, \lambda+, \ldots.. \lambda_n)$

Modal & Spectral Matrix:-

The matrix P in the above result which diagonalise the square matrix A is called the Modal matrix and the resulting diagonal D is called is known as spectral matrix.

Note:- If the eigen values of an nxn matrix are all district then it is always similar to a diagonal matrix.

Calculation of power of a matrix:-

Let A be the Square matrix. Then a non-singular matrix P can be found

$\rightarrow D = P^{-1}AP$

$D^2 = (P^{-1}AP)(P^{-1}AP) = P^{-1}A(PP^{-1})AP = P^{-1}A^2P$

$D^3 = P^{-1}A^3P$

$D^n = P^{-1}A^nP$ …………….. (1)

Premultiply (1) by P & Post multiply by P^{-1}

$PD^nP^{-1} = P(P^{-1}A^nP)P^{-1} = (PP^{-1})A^n(PP^{-1}) = A^n$

$\Rightarrow A^n = PD^nP^{-1}$

$$A^n = P\begin{bmatrix} \lambda^n & 0 & 0 & . & 0 \\ 0 & \lambda 2^n & 0 & . & 0 \\ 0 & 0 & \lambda 3^n & . & 0 \\ . & . & . & . & . \\ 0 & 0 & 0 & . & \lambda n^n \end{bmatrix} P^{-1}$$

1) Diagonalize the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ -4 & 4 & 3 \end{bmatrix}$ find A^4 (or) find a matrix P which transform the matrix

$A = \begin{bmatrix} 1 & 1 & 4 \\ 0 & 2 & 5 \\ -4 & 4 & 3 \end{bmatrix}$ to diagonal form Hence calculate A^4 and find the eigen value A^{-1}

Sol:- $A = \begin{bmatrix} 1 & 1 & 4 \\ 0 & 2 & 5 \\ -4 & 4 & 3 \end{bmatrix}$ Characteristic Equaltion $|A-\lambda I| = 0$

$$\begin{bmatrix} 1-\lambda & 1 & 1 \\ 0 & 2-\lambda & 1 \\ -4 & 4 & 3-\lambda \end{bmatrix} = 0$$

$(1-\lambda)(2-\lambda)(3-\lambda) = 0$

$\lambda = 1,2,3$

Characteristic vector corresponding to $\lambda = 1$

$[A-\lambda I] = 0$

$[A-I] = 0$

$$\begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ -4 & 4 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Y+z = 0; $\Rightarrow$ y = -z

y+z = 0; let z = k

$-4x+4y+2z = 0$ $\qquad$ $y = -k$

$X = -k/2$

Eigen vector $x_1 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2k/2 \\ k \\ k \end{bmatrix} = -k/2 \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$

$\therefore x_1 = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$

Characteristic vector corresponding to $\lambda = 2$

$[A-\lambda I]x = 0; [A-2I]x = 0$

$$\begin{bmatrix} -2 & 1 & 1 \\ 0 & 0 & 1 \\ -4 & 4 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \to R_3-4R_1$ $\begin{bmatrix} -1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z` \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$R_3 \to R_3-3R_2$ $\begin{bmatrix} -1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

r=2, n = 3, n-r = 3-2 = 1 we have to give one variables to the one arbitrary constant.

$-x+y+z=0; z = 0$

Then we take x (or) y = y

$\therefore y = k$

$-x+k+0 = 0$

$x=k, y=k, z=0$

$\Rightarrow x_2 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} k \\ k \\ 0 \end{bmatrix} = k \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$\therefore$ Eigen value of A^{-1}

Characteristic vector corresponding to $\lambda = 3$

$$\begin{bmatrix} -2 & 1 & 1 \\ 0 & -1 & 1 \\ -4 & 4 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \to R_3-4R_1$ $\begin{bmatrix} -2 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 2 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$R_3 \to R_3-2R_2$ $\begin{bmatrix} -2 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

r=2, n=3, n-r=3-2 = 1

$-2x+y+7 = 0$

$-y+z=0$

Let $z = k$

$-y=-z=-k \Rightarrow y = k$

$-2z = -y-z = -k-k$

$-2x=-2k \Rightarrow x = k$

Eigen vector x3 = $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \left[\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right] k = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$P = [x_1 \ \ x_2 \ \ x_3]$

Model matrix = p = $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ -2 & 0 & 1 \end{bmatrix}$

P-1 = adjp/detp = $\begin{bmatrix} -1 & 1 & 0 \\ 4 & -3 & -1 \\ -2 & 2 & 1 \end{bmatrix}$

$P^{-1}AP = \begin{bmatrix} -1 & 1 & 0 \\ 4 & -3 & -1 \\ -2 & 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ -4 & 4 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ -2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ =D

2) P-1AP = D = $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ = Diagonalization

Power of a matrix $A^n = PD^nP^{-1}$; $A^4=PD^4P^{-1}$

$A4 = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 81 \end{bmatrix} \begin{bmatrix} -2 & 1 & 0 \\ 4 & -3 & -1 \\ -2 & 2 & -1 \end{bmatrix}$

$= \begin{bmatrix} -99 & 115 & 65 \\ -100 & 116 & 65 \\ -160 & -160 & 81 \end{bmatrix}$

Eigen value of $A^{-1} = 1/\lambda = 1/1, 1/2, 1/3$

2. find the diagonal matrix that will diagonaize the real symmetric matrix A= $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$

Also find the resulting diagonal matrix. (or) Diagonalize the matrix A= $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$

Sol:- A= $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$ Characteristic Equation $|A-\lambda I| = 0$

$\begin{bmatrix} 1-\lambda & 2 & 3 \\ 2 & 4-\lambda & 6 \\ 3 & 6 & 9-\lambda \end{bmatrix} = 0$

$\Rightarrow \lambda(\lambda^2-14\lambda) = 0$

$\lambda = 0, 0, 14$ Eigen roots $\lambda = 0, 0, 14$

Eigen vector corresponding to $\lambda = 14$

$[A-14I]x = 0$

$$\begin{bmatrix} -13 & 2 & 3 \\ 2 & -10 & 6 \\ 3 & 6 & -5 \end{bmatrix}\begin{bmatrix} x1 \\ x2 \\ x3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_1 = 1, x_2 = 2, x_3 = 3$

Eigen vector $x_1 = \begin{bmatrix} x1 \\ x2 \\ x3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

To the Eigen Vector corresponding to $\lambda = 0$

$[A-\lambda I]x =$

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}\begin{bmatrix} x1 \\ x2 \\ x3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2\text{-}2r_1$; $R_3 \to R_3\text{-}3R_1$ $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x1 \\ x2 \\ x3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

r=1, n = 3, n-r = 3 – 1 = 2

let $x2 = k_1$, $x_3 = k_2$

$x_1+2x_2+3x_3 = 0$

$x_1=-2k_1-3k_2$

$x_2=k_1$

$x_3=k_2$

Eigen vector = $\begin{bmatrix} -2k1-3k2 \\ k1 \\ k2 \end{bmatrix} = k_1\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + k_2\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$

$\therefore$ $x_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $x_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$, $x_3 = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$

Normalised Model matrix = p = $[x1 \quad x2 \quad x3] = \begin{bmatrix} 1 & -2 & -3 \\ 2 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$

$$P = \left[\frac{x1}{||x1||} \frac{x2}{||x2||} \frac{x3}{||x3||}\right] = \begin{bmatrix} 1/\sqrt{14} & -2/\sqrt{5} & -3/\sqrt{10} \\ 2/\sqrt{14} & 1/\sqrt{5} & 0 \\ 3/\sqrt{14} & 0 & 1/\sqrt{10} \end{bmatrix}$$

$$\Rightarrow P^{-1} = P^T = \begin{bmatrix} 1/\sqrt{14} & 2/\sqrt{14} & 3/\sqrt{14} \\ -2/\sqrt{5} & 1/\sqrt{5} & 0 \\ -3/\sqrt{10} & 0 & 1/\sqrt{10} \end{bmatrix}$$

$$P^{-1}AP=P^TAP= \begin{bmatrix} 1/\sqrt{14} & 2/\sqrt{14} & 3/\sqrt{14} \\ -2/\sqrt{5} & 1/\sqrt{5} & 0 \\ -3/\sqrt{10} & 0 & 1/\sqrt{10} \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}\begin{bmatrix} 1/\sqrt{14} & 2/\sqrt{5} & -3/\sqrt{10} \\ 2/\sqrt{14} & 1/\sqrt{5} & 0 \\ 3/\sqrt{14} & 0 & 1/\sqrt{10} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 14 \end{bmatrix} \Rightarrow P^{-1}AP = P^TAP = D$$

\ A is reduced to diagonal form by orthogonal reduction.

Exercise problems:

1. Diagonalize the matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix}$ by orthogonal reduction (or) Diagonalize the matrix.

2) Determine the diagonal matrix orthogonally similar to the following symmetric matrix

$A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$

3) Determine the diagonal matrix orthogonally similar to the following symmetrix matrix

$A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

4) Diagonalize the matrix $A = \begin{bmatrix} 8 & -8 & -2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix}$

5) Find a matrix P which transorm the matrix $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$ to diagonal form.

Hence calculate A^4 (or) Diagonalize the matrix $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$

6) Prove that the matrix $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is not diagonalizable.

7) S.T. the matrix $A = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{bmatrix}$ cannot be diagonalized.

Quadratic forms

Quadratic form:-

A homogeneous expression of the second degree in any number of variables is called a quadratic form. An expression of the form $Q = x^TAx = \sum_{i=1}^{n} . \sum_{j=1}^{n} .$ aij x i x j where aij's are constants is called quadratic form in n variables $x_1, x_2, \ldots\ldots x_n$. If the constants aij's are real numbers it is called a real quadratic form. $[x_1, x_2, \ldots\ldots x_n]$

$Q = x^TAx$ Ex-1) $3x^2+5xy-2y^2$ is a quadratic form in two variables x & y

2) $2x^2+3y^2-4z^2+2xy-3yz+5zx$ is a quadratic form of 3 variables x,y,& z

Symmetric Matrix :-

$Q = X^T AX$ is a quadratic form where A is known as real symmetric matrix.

$$A = \text{symmetric Matrix} = \begin{bmatrix} coeff.of\ x1^2 & \frac{1}{2}coeff.of\ x1x2 & \frac{1}{2}coeff\ of\ x1x2 \\ \frac{1}{2}coeff\ of\ x1x2 & coeff.\ of\ x2^2 & \frac{1}{2}coeff\ of\ x2x3 \\ \frac{1}{2}coeff\ of\ x1x3 & \frac{1}{2}coef\ of\ x2x3 & coeff\ of\ x3^2 \end{bmatrix}$$

Eg 1:- Write the symmetric matrix of the quadratic form x_1^2-$+6x_1x_2+5x_2^2$

Sol:- Symmetric matrix of the quadratic form $x1^2+6x_1x_2+5x_2^2$

Sol:- A Symmetrix matrix $= \begin{matrix} x1 \\ x2 \end{matrix} \begin{bmatrix} 1 & \frac{6}{2} \\ \frac{6}{2} & 5 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 3 & 5 \end{bmatrix}$

2) Find the symmetric matrix of the quadratic form $x_1^2+2x_2^2+4x_2x_3+x_3x_4$

Sol:-

$$\begin{matrix} & x1 & x2 & x3 & x4 \end{matrix}$$

$$= \begin{matrix} x1 \\ x2 \\ x3 \\ x4 \end{matrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 2 & 0 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & 0 \end{bmatrix}$$

3) find the quadratic form of the given symmetric matrix $A \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$

Sol:- Quadratic form $= X^T AX = [x\ \ y\ z] \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$= ax^2+by^2+cz^2+2hxy+2gxz+2fyz$

Exercise Problems:-

Write the Symmetrix matrix of the following quadratic forms

1. $x_1^2+2x_2^2-7x3^2-4x_1x_2+8x_1x_3$
2. $x_1^2+2x_2^2-7x_3^2-4x_1x_2+8x_1x_3+5x_2x_3$
3. $2x_1x_2+6x_1x_3-4x_2x_3$
4. $x^2+2y^2+3z^2+4xy+5yz+6zx$
5. $x^2+y^2+z^2+2xt+2yz+3zt+4t^2$
6. Obtain the quadratic form of the following Matrices.

1) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 3 \\ 3 & 3 & 1 \end{bmatrix}$ 2) $\begin{bmatrix} 2 & 1 & 5 \\ 1 & 3 & 4 \\ 5 & 4 & 5 \end{bmatrix}$ 3) $\begin{bmatrix} 1 & 3 \\ 3 & 2 \end{bmatrix}$

4) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 5 & 6 & 7 \\ 3 & 6 & 0 & 1 \\ 4 & 7 & 1 & 2 \end{bmatrix}$ 5) $\begin{bmatrix} 3 & 4 \\ 4 & 5 \end{bmatrix}$ 6) $\begin{bmatrix} 1 & 0 & 3 \\ 0 & 2 & 5 \\ 3 & 5 & 4 \end{bmatrix}$

Canonical form

The conanical form of a quadratic form x^TAx is y^TDy (or) $\lambda_1 y_1^2+\lambda_2 y_2^2+\ldots..+\lambda_n y_n^2$

This form is also known as the sum of the squares form or principal axes form

Canonical form = y^TDy = $[y_1 y_2\ y_3] \begin{bmatrix} \lambda 1 & 0 & 0 \\ 0 & \lambda 2 & 0 \\ 0 & 0 & \lambda 3 \end{bmatrix} \begin{bmatrix} y1 \\ y2 \\ y3 \end{bmatrix} = \lambda_1 y_1^2+\lambda_2 y_2^2+\lambda_3 y_3^2$

Reduction of Quadratic form to canonical form by Linear Transformation.

Consider a quadratic form in n variables

x^TAx and a non singular linear transformation $x = Py$ then $x^T= [Py]^T = y^TP^T$

$x^TAx = y^TP^TAPy = y^T(P^TAP)y = y^TDy$ where $D = P^TAP$

$\Rightarrow x^TAx = y^TDy$

Thus, the quadratic form x^TAx is reduced to the canonical form y^TDy. The diagonal Matrix D and matrix A and called Congruent matrices.

Reduction of Quadratic

Nature of the Quadratic form

The quadratic form x^TAx in n variables is said to be

1) Positive definite:-

If r = n & s = n (or) if all the eigen values are +ve.

2) Negative definite:-

If r = n & s = 0 (or) if all the eigen values are –ve.

3) Positive semidefinite (or) semipositive:-

If r<n & s=r (or) if all the eigen values of A≥0 & atleast one eigen value is zero.

4) semi negative:-

If r<n & s = 0 (or) if all the eigen values of A≤0 & atlease one eigen value is zero.

5) Indefinite:-

In all other cases (or) some are positive, -ve.

→**Index of a real quadratic form**

The number of positive terms in canonical form (or) normal form of a quadratic form is known as the index. It is denoted by ‘s’

Signature of a quadratic form

If r is the rank of a quadratic form & s is the number of positive terms in its normal form, then ∃ number of positive terms over the number of negative terms in a normal form of x^TAx . ∴ Signature = [+ve terms] – [-ve terms]

Note:- Signature = 2s-r

Where s→index

r→rank = no. of non zero rows.

Short Answer question:-

1) Find the nature, rank, Index of a quadratic form $2x^2+2y^2+2z^2+2yz$

Sol :- $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$

$|A-\lambda I| = 0 \Rightarrow \begin{bmatrix} 2-\lambda & 0 & 0 \\ 0 & 2-\lambda & 1 \\ 0 & 1 & 2-\lambda \end{bmatrix} = 0$

$\lambda = 1, 2, 3$

Nature ;- all th eigen values are +ve

⇒ positive definite

Rank:- r = 3

Index : S = no. of positive terms = 3

Signature: - [+ve terms] – [-ve terms] = 3 – 0 = 3

Discuss the nature of the given quadratic form

1) $x_1^2+4x_2^2+x_3^2-4x_1x_2+2x_1x_3-4x_2x_3$

2) $x^2+4xy+6xz-y^2+2yz+4z^2$

Reduction of Quadratic form to canonical form by orthogonal reduction:

1) Write the coefficient matrix A associated with the given quadratic form

2) A = symmetric Matrix = [ ]

3) Find the eigen values & eigen vectors.

4) Model Matrix $P = [x_1\ x_2\ x_3]$

5) Normalized model matrix $\acute{P} = \left[\frac{x1}{||x1||}\ \ \frac{x2}{||x2||}\ \ \frac{x3}{||x3||}\right]$

6) Find P^{-1}; $P^{-1} = P^T$

7) $P^{-1}AP = P^TAP = D = \begin{bmatrix} \lambda 1 & 0 & 0 \\ 0 & \lambda 2 & 0 \\ 0 & 0 & \lambda 3 \end{bmatrix}$

8) Canoniclal form $= y^TDy = [y_1\ y_2\ y_3] \begin{bmatrix} \lambda1 & 0 & 0 \\ 0 & \lambda2 & 0 \\ 0 & 0 & \lambda3 \end{bmatrix} \begin{bmatrix} y1 \\ y2 \\ y3 \end{bmatrix}$

$= \lambda_1 y_1^2+\lambda_2 y_2^2+\lambda_3 y_3^2$

9) Linear transformation is x = Py,

1. Reduce the quadratic form $3x^2+2y^2+3z^2-2xy-2yz$ to the normal form by orthogonal transformation . Also write the rank, Index, nature and signature.

Sol:- given quadratic form $3x^2+2y^2+3z^2-2xy-2yz$ $A = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{bmatrix}$

Characteristic equation is $|A-\lambda I| = 0$

$$\begin{bmatrix} 3-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 3-\lambda \end{bmatrix} = 0$$

λ = 3, 1, 4; eigen values λ = 3, 1, 4

Eigen vectors $x_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, $x_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, $x_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

$P = [x_1\ x_2\ x_3] \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & -1 \\ -1 & 1 & 1 \end{bmatrix}$

P = normalized model matrix $P=[e_1\ e_2\ e_3] = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{6} \\ 0 & 2/\sqrt{6} & -1/\sqrt{6} \\ -1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{6} \end{bmatrix}$

P is orthogonal $P^{-1} = P^T = \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/\sqrt{6} & 2/\sqrt{6} & 1/\sqrt{6} \\ 1/\sqrt{6} & -1/\sqrt{6} & 1/\sqrt{6} \end{bmatrix}$

$$P^{-1}AP = P^TAP = \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/\sqrt{6} & 2/\sqrt{6} & 1/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix} \begin{bmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & -1/\sqrt{3} \\ -1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \end{bmatrix}$$

$= \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$ = D & the quadratic form will be reduced to the normal form

<u>Canonical form = y^TDy</u>

$= [y_1\ \ y_2\ \ y_3] \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} y1 \\ y2 \\ y3 \end{bmatrix}$

$= 3y_1^2+y_2^2+4y_3^2$

Index :- No.of positive terms = S = 3

Rank:- r = 3

Nature:- all eigen values are +ve = S = 3

Signature:- = [no of +ve terms] – [no. of –ve terms]

= 3-0 = 3

Orthogonal transformation is x = Py

$$x = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & -1/\sqrt{3} \\ -1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \end{bmatrix} \begin{bmatrix} y1 \\ y2 \\ y3 \end{bmatrix}$$

$x = y_1/\sqrt{2} + 1/\sqrt{6}y_2 + 1/\sqrt{3}\, y_3$

$y = 2/\sqrt{6}y2 - 1/\sqrt{3}\, y_3$

$z = -1/\sqrt{2}y1 + 1/\sqrt{6}y2 + 1/\sqrt{3}\, y_3$

Exercise:

Reduce the Quadratic form to canonical form by orthogonal Reduction. And write the transformation, nature index, rank, signature

1) $2x^2+2y^2+2z^2-2xy+2zx-2yz$

2) $x_1^2+3x_2^2+3x_3^2-2x_2x_3$

3) $3x^2+5y^2+3z^2-2yz+2zx-2xy$

4) $6x^2+3y^2+3z^2-2yz+4zx-4xy$

2) for the matrix $A = \begin{bmatrix} 1 & 2 & -3 \\ 0 & 3 & 2 \\ 0 & 0 & -2 \end{bmatrix}$ find the eigen values of $3A^3+5A^2-6A+2I$

Sol:- $A=\begin{bmatrix} 1 & 2 & -3 \\ 0 & 3 & 2 \\ 0 & 0 & -2 \end{bmatrix}$ characteristic egn is $|A-\lambda I| = 0$

$$\begin{bmatrix} 1-\lambda & 2 & -3 \\ 0 & 3-\lambda & 2 \\ 0 & 0 & -2-\lambda \end{bmatrix} = 0$$

$(1-\lambda)(3-\lambda)(-2-\lambda) = 0;\ \lambda=1,3,-2$

λ is the Eigen value of A & f(A) is a polynomial in A, then the eigen value of f(A) is $f(\lambda)$

$f(A) = 3A^3+5A^2-6A+2I$

Then the eigen value of f(A) are

$f(1) = 3(1)^3+5(1)^2-6(1)+2 = 4$

$f(3) = 3(3)^3+5(3)^2-6(3)+2(1) = 110$

$f(-2) = 3(-2)^3+5(-2)^2-6(-2)+2(1) = 10$

Thus the Eigen value of $3A^3+5A^2-6A+2I$ are 4, 110, 10

→P.T. the matrix A = $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is not diagonalizable.

Sol:- The characteristic equation is |A-λI| = 0

$$\begin{bmatrix} -\lambda & 1 \\ 0 & -\lambda \end{bmatrix} = 0 \Rightarrow \lambda^2=0$$

$$\lambda = 0,0$$

λ= 0, The characteristic vector. [A-λI]x = 0

$= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x1 \\ x2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$x_2=0$, $x_1=k$

The characteristic vector is $\begin{bmatrix} k \\ 0 \end{bmatrix} = K \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

The given matrix has only one i.j. charactestic vector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ corresponding to repeated characteristic value '0'

The matrix is not diagonalizable

Note: A is nilpolent matrix ⇒ A is not diagonalised.

Eg:- Determine the eigen values & eigen vectors of B = 2A2-1/2A+3I where A= $\begin{bmatrix} 8 & -4 \\ 2 & 2 \end{bmatrix}$

Sol:- A = $\begin{bmatrix} 8 & -4 \\ 2 & 2 \end{bmatrix}$ characteristic equation is

|A-λI| = 0

$\begin{bmatrix} 8-\lambda & -4 \\ 2 & 2-\lambda \end{bmatrix} = 0 \Rightarrow (8-\lambda)(2-\lambda)+8 = 0$

$$16-8\lambda-2\lambda+\lambda^2+8=0$$

$$\lambda^2-10\lambda+24=0$$

$$\lambda^2-6\lambda-4\lambda+24=0$$

$$\lambda(\lambda-6) - 4(\lambda-6)=0$$

$$(\lambda-6)(\lambda-4)=0$$

$$\lambda= 6, 4$$

$B= 2A^2- ½ A+3I$

λ is the eigen value of A

Then the eigen value of B is

$B= 2(6)^2 - ½ (6)+3$, $B = 2(4)^2 - ½ (4)+3 = 72, 33$

Eigen value of B is 33,72

$B = 2A^2 - ½ A + 3I= \begin{bmatrix} 112 & -80 \\ 40 & -8 \end{bmatrix} - \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 111 & -78 \\ 39 & -6 \end{bmatrix}$

Characteristic Equation |B-λI| = 0

$\begin{bmatrix} 111-\lambda & -78 \\ 39 & -6-\lambda \end{bmatrix} = 0 \Rightarrow \lambda^2+105-2376 = 0$

$\lambda = 33, 72$

Eigen value of B are 33 & 72

λ=33, the eigen vector of B is given by [B-33I]x = 0

$\begin{bmatrix} 78 & -78 \\ 39 & -39 \end{bmatrix}\begin{bmatrix} x1 \\ x2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow x = 1, x_2 = 1$

$\lambda=33, x1= \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

λ=72, the eigen vector of B is given by [B-72I]x = 0

$\begin{bmatrix} 39 & -78 \\ 39 & -78 \end{bmatrix}\begin{bmatrix} x1 \\ x2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow x_2 = 1, x_1 = 2$

∴The eigen vector for λ=72, x2= $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

1) Find the inverse transformation of $y_1=2x_1+x_2+x_3$, $y_2 = x_1+x_2+2x_3$, $y_3 = x_1-2x_3$

Sol: The given transformation can be written as

$\begin{bmatrix} y1 \\ y2 \\ y3 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 0 & -2 \end{bmatrix}\begin{bmatrix} x1 \\ x2 \\ x3 \end{bmatrix}$

Y=Ax

$|A| = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 0 & -2 \end{bmatrix} = -1 \neq 0$

Thus the matrix A is non-singular and hence the transformation is regular. The inverse transformation is given by $x=A^{-1}y$

$\begin{bmatrix} x1 \\ x2 \\ x3 \end{bmatrix} = \begin{bmatrix} 2 & -2 & -1 \\ -4 & 5 & 3 \\ 1 & -1 & -1 \end{bmatrix}\begin{bmatrix} y1 \\ y2 \\ y3 \end{bmatrix}$

$x_1= 2y_1-2y_2-y_3$

$x_2=-4y_1+5y_2+3y_3$

$x_3= y_1-y_2-y_3$

2) S.T. the transformation $y_1=x_1\cos\theta = x_2\text{sing}\theta$, $y_2 = -x1\sin\theta+x_2\cos\theta$ is orthogonal.

Sol:- The given transformation can be written as Y=Ax

$Y = \begin{bmatrix} y1 \\ y2 \end{bmatrix} \quad A = \begin{bmatrix} c\ \theta & i\ \theta \\ -i\ \theta & c\ \theta \end{bmatrix}, \quad x = \begin{bmatrix} x1 \\ x2 \end{bmatrix}$

Here the matrix of transformation is $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$, $A^{-1} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = A^T$

the transformation is orthogonal.

Cayley – Hamilton Theorem

Theorem:- Every square matrix statisfies its own characteristic equation.

Applications of cayley – Hamilton Theorem

The important applications of Cayley – Hamilton theorem are

1) To find the inverse of a matrix

2) To find higher powers of a matrix.

1) If $A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & -2 \\ 2 & -2 & -1 \end{bmatrix}$ verify cayley – Hamilton theorem

Find A^{-1} & A^4 using cayley – Hamilton theorem.

Sol: $A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & -2 \\ 2 & -2 & -1 \end{bmatrix}$ Characteristic Equation $|A-\lambda I| = 0$

$\begin{bmatrix} 1-\lambda & 2 & -1 \\ 2 & 1-\lambda & -2 \\ 2 & -2 & -1-\lambda \end{bmatrix}$ $x^3-3\lambda^2-3\lambda+9=0$

By cayley – Hamilton theorem, matrix A should satisfy its characterstic Equation.

i.e., $A^3-3A^2-3A+9I = 0$

$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & -2 \\ 2 & -2 & +1 \end{bmatrix}$

$A^2=A.A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & -2 \\ 2 & -2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 6 & -6 \\ 0 & 9 & -6 \\ 0 & 0 & 3 \end{bmatrix}$

$A^3=A^2.A = \begin{bmatrix} 3 & 6 & -6 \\ 0 & 9 & -6 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 24 & -21 \\ 6 & 21 & -24 \\ 6 & -6 & 3 \end{bmatrix}$

$A^3-3A^2-3A+9I =$

$\begin{bmatrix} 3 & 24 & -21 \\ 6 & 21 & -24 \\ 6 & -6 & 3 \end{bmatrix} -3 \begin{bmatrix} 3 & 6 & -6 \\ 0 & 9 & -6 \\ 0 & 0 & 3 \end{bmatrix} -3 \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix} +9 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$A^3-3A^2-3A+9I = 0$

Hence cayley – Hamilton is verified.

To find A^{-1}:-

Multiplying equation (1) with A^{-1} on b/s

$A^{-1}[A^3-3A^2-3A+9]=0$

$A^2-3A-3AI+9A^{-1} = 0$

$9A^{-1} = 3A+3I-A^2$

$A^{-1} = \frac{1}{9}[3A+3I-A^2]$

$$A^{-1} = \frac{1}{9}[3A+3I-A^2] = \frac{1}{9}\left\{\begin{bmatrix} 3 & 6 & -3 \\ 6 & 3 & -6 \\ 6 & -6 & 3 \end{bmatrix} + \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} - \begin{bmatrix} 3 & 6 & -6 \\ 0 & 9 & -6 \\ 0 & 0 & 3 \end{bmatrix}\right\}$$

$$= \begin{bmatrix} \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{2}{3} & \frac{-1}{3} & 0 \\ \frac{2}{3} & \frac{-2}{3} & \frac{1}{3} \end{bmatrix}$$

Find A^4 :-

Multiplying with A

$A[A^3-3A^2-3A+9I] = 0$

$A^4 = 3A^3+3A^2-9A$

$$= 3\begin{bmatrix} 3 & 24 & -21 \\ 6 & 21 & -24 \\ 6 & -6 & 3 \end{bmatrix} + 3\begin{bmatrix} 3 & 6 & -6 \\ 0 & 9 & -6 \\ 0 & 0 & 3 \end{bmatrix} - 9\begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 9 & 72 & -72 \\ 0 & 81 & -72 \\ 0 & 0 & 9 \end{bmatrix}$$

1) Show that the matrix satisfies its characteristic Equation Find A^{-1}& A^4 (or) verify cayley Hamilton Theorem. Find A^{-1}& A^4

1) $A = \begin{bmatrix} 1 & -2 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & 2 \end{bmatrix}$

2) $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$

3) $A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & -1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$

4) $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$

5) $A = \begin{bmatrix} 3 & 4 & 1 \\ 2 & 1 & 6 \\ -1 & 4 & 7 \end{bmatrix}$

1) using cayley – Hamilton theorm. Find $A^{8,}$ If $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

Sol:- $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ Characteristic Equation

$|A-\lambda I| = 0$

$\begin{bmatrix} 1-\lambda & 2 \\ 2 & -1-\lambda \end{bmatrix} = 0$

$\lambda^2 - 5 = 0$

By cayley – Hamilton Theorem. Every square matrix satisfied its characteristic equation.

$A^2 - 5 = 0$

$A^2 = 5I$

$A^8 = A^2.A^2.A^2 = [5I].[5I].[5I]$

$A^8 = 625I$

2) If $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$, find the value the matrix $A^8-5A^7+7A^6-3A^5+A^4-5A^3+8A^2-2A+I$

Sol: The characteristic Equation $|A-\lambda I| = 0$

$\begin{bmatrix} 2-\lambda & 1 & -1 \\ 0 & 1-\lambda & 0 \\ 1 & 1 & 2-\lambda \end{bmatrix} = 0$

$x^3-5\lambda^2-7\lambda-3=0$ By Cayley Hamilton theorm

$A^3-5A^2+7A-3I=0$

We can rewrite the given expression as $A^5[A^3-5A^2+7A-3I] + A[A^3-5A^2+7A-3I]$

$A^8-5A^7+7A^6-3A^5+A^4-5A^3+8A^2-2A+I$

$= A^5[A^3-5A^2+7A-3I] + A[A^3-5A^2+8A-2I]=I$

$= A^5(0) + A[A^3-5A^2+7A-3I] + A^2+A+I=I$

$A[A^3-5A^2+7A-3I] + (A+I)]+I$

$= A^2+A+I$

But $A^2+A+I = \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 5 & 5 \\ 0 & 3 & 0 \\ 5 & 5 & 8 \end{bmatrix}$

Exercise:

1) If $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$ write $2A^5-3A^4+A^2-4I$ as a linear polynomial in A

Sol:- $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$ $|A-\lambda I| = 0$

$\begin{bmatrix} 3-\lambda & 1 \\ -1 & 2-\lambda \end{bmatrix} = 0 \Rightarrow \lambda 2-5\lambda+7 = 0$

By cayley – Hamilton Theorm,

A must satisfy its characteristic equation.

$A^2-5A+7I = 0$

$A^2= 5A-7I$

$A^3 = 5A^2-7A$

$A^4=5A^3-7A^2$

$A^5=5A^4-7A^3$

$2A^5-3A^4+A^2-4I$

$=2[5A^4-7A^3]-3[5A^3-7A^2]+[5A-7I]-4I$

$= 7A^4-14A^3+A^2-4I$

$= 7[5A^3-7A^2]-14A^3+A^2-4I$

$= 21A^3-48A^2-4I$

$= 21(5A^2-7A) -48A^2-4I$

$= 57A^2-147A-4I$

$= 57(5A-7I) -147A-4I$

$= 138A-403I$ which is a linear poly in A

Unit – II(Important questions)

1. Find all the eigen values of $A^2+3A-2I$, If $A = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$ 2 Marks
2. Find the nature, index, signature of the quadratic form $3x^2+5y^2+3z^2$. 3Marks
3. Find the Eigenvalues & Eigenvectors of the matrix $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$ 5 Marks
4. Verify cayley – Hamilton theorem for the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$ Express $B= A^8-11A^7-4A^6+A^5+A^4-11A^3-3A^2+2A+I$ as a quadratic poly in A 5 Marks
5. Diagonalize the Matrix A = $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ -4 & 4 & 3 \end{bmatrix}$ hence find A^4 5 Marks
6. Reduce the Q.F. to C.F. C.F. Hence find its nature $x^2+y^2+z^2-2xy+4xz+4yz$ 5 Marks
7. Find the sum & product of the Eigen values of the matrix A $\begin{bmatrix} 2 & 5 & 7 \\ 1 & 4 & 6 \\ 2 & -2 & 3 \end{bmatrix}$ 2Marks
8. Write the quadratic form Corresponding to the matrix $A = \begin{bmatrix} 1 & 5 & 7 \\ 5 & 4 & 6 \\ 7 & 6 & 3 \end{bmatrix}$ 3 Marks
9. Find the eigen values $5A^2-2A^2+7A-3A^{-1}+I$ if $A = \begin{bmatrix} -3 & -7 & -5 \\ 2 & 4 & 3 \\ 1 & 2 & 2 \end{bmatrix}$ 5 Marks
10. Using cayley – Hamilton Then find A^{-1}& A^{-2} if $A = \begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -4 & -3 \end{bmatrix}$ 5 Marks

11. Reduce the Q.form $8x^2+7y^2+3z^2+12xy+4xz+8yz$ to canonical form and find rank, nature, index & signature 10 Marks

Properties of Eigen Values:

Theorm 1: The sum of the eigen values of a square matrix is equal to its trace and product of the eigen values is equal to its determinant.

Proof: Characterristic equation of A is . $|A-\lambda I|=0$

i.e, $\begin{bmatrix} a_{11}-\lambda & a_{12} & L & a_{1n} \\ a_{21} & a_{22}-\lambda & L & a_{2n} \\ L & L & L & L \\ a_{n1} & a_{n2} & L & a_{nn}-\lambda \end{bmatrix}$ $expanding\ this\ we\ get$

$(a_{11}-\lambda)(a_{22}-\lambda) L\ (a_{nn}-\lambda) - a_{12}\,(a\ polynomial\ of\ degree\ n-2)$

$+\ a_{13}\ (a\ polynomial\ of\ degree\ n+2) + \ldots + = 0$

$$\Longrightarrow (-1)^n[\lambda^n - (a_{11}+a_{22}+\ldots.+a_{nn})\lambda^{n-1} + a\ polynomila\ of\ degree\ (n-2)]$$

$$(-1)^n\,\lambda^n + (-1)^{n+1}(Trave\ A)\lambda^{n-1} + +a\ polynomial\ of\ degree\ (n-2)in\ \lambda = 0$$

$$If\ \lambda_1, \lambda_2 \ldots\ldots \lambda_n\ are\ the\ roots\ of\ this\ equation$$

$$sum\ of\ the\ roots = \frac{(-1)^{n+1}Tr(A)}{(-1)^n} = Tr(A)$$

$$u\ r\ t\ her\ |\quad -\lambda\ | = (-1)^n\lambda^n + \ .+a_0$$

$$put\ \lambda = 0\ then\ |A| = a_0$$

$$(-1)^n\lambda^n + a_{n-1}\lambda^{n-1} + a_{n-2}\lambda^{n-2} + \ \ldots.+a_0 = 0$$

$$Product\ of\ the\ roots = \frac{(-1)^n\,a_0}{(-1)^n} = a_0$$

$$but\ a_0 = |A| = \det A$$

$$Hence\ the\ result$$

Theorm 2: If λ is an eigen value of A corresponding to the eigen vector X, then λ^n is eigen value A^n corresponding to the eigen vector X.

Proof: Since λ is an eigen value of A corresponding to the eigen value X, we have

$AX= \lambda X$ ----------(1)

Pre multiply (1) by A, $A(AX) = A(\lambda X)$

$(AA)X = \lambda(AX)$

$A^2X= \lambda(\lambda X)$

$A^2X= \lambda^2X$

Λ^2 is eigen value of A^2 with X itself as the corresponding eigen vector. Thus the theorm is true for n=2

let we assume it is true for n = k

i.e,, $A^K X = \lambda^K X$------------(2)

Premultiplying (2) by A, we get

$A(A^k X) = A(\lambda^K X)$

$(AA^K)X = \lambda^K(AX) = \lambda^K(\lambda X)$

$A^{K+1}X = \lambda^{K+1}X$

λ^{K+1} is eigen value of A^{K+1} with X itself as the corresponding eigen vector.

Thus, by M.I., λ^n is an eigen value of A^n

Theorm 3: A Square matrix A and its transpose A^T have the same eigen values.

Proof: We have $(A-\lambda I)^T = A^T - \lambda I^T$

$= A^T - \lambda I$

$|(A-\lambda I)^T| = |A^T - \lambda I|$ (or)

$|A-\lambda I| = |A^T - \lambda I|$

$|A-\lambda I| = 0$ if and only if $|A^T - \lambda I| = 0$

λ is eigen value of A if and only if λ is eigen value of A^T

Hence the theorm

Theorrm 4: If A and B are n-rowed square matrices and If A is invertible show that $A^{-1}B$ and BA^{-1} have same eigen values.

Proof: Given A is invertile

i.e, A^{-1} exist

w e know that if A and P are the square matrices of order n such that P is non-singular then A and $P^{-1}AP$ hence the same eigen values.

Taking $A=BA^{-1}$ and P=A, we have

BA^{-1} and $A^{-1}(BA^{-1})A$ have the same eigen value

BA^{-1} and $(A^{-1}B)(A^{-1}A)$ have the same eigen values

BA^{-1} and $(A^{-1}B)I$ have the same eigen values

BA^{-1} and $A^{-1}B$ have the same eigen values

Theorm 5: If $\lambda 1, \lambda 2, \ldots\ldots \lambda n$ are the eigen values of a matrix A then $k\lambda_1, k\lambda_2, \ldots\ldots k\lambda_n$ are the eigen value of the matrix KA, where K is a non-zero scalar.

Proof: Let A be a square matrix of order n. Then $|KA-\lambda KI| = |K(A-\lambda I| = K^n|A-\lambda I|$

Since $K \neq 0$, therefore $|KA - \lambda KI| = 0 \implies |A - \lambda I| = 0$

$i.e, K\lambda$ is an eigen value of KA $\iff$ if λ is an eigen value of A

Thus $k\lambda_1, k\lambda_2 \ldots k\lambda_n$ *are the eigen values of the matrix KA*

$\iff \lambda_1, \lambda_2 \ldots \lambda_n$ are the eigen values of the matrix A

<u>Theorm 6</u>: If λ is an eigen value of the matrix. Then λ+K is an eigen value of the matrix A+KI

Proof: Let λ be an eigen value of A and X the corresponding eigen vector. Then by definition AX= λX

Now (A+KI)X = $(\lambda + KI)X$

$= \lambda X + KX$

$=(\lambda + K)X$

$\lambda + K$ is an eigen value of the matrix $A + KI$

<u>Theorm 7:</u> If $\lambda_1, \lambda_2 \ldots \lambda_n$ are the eigen values of A the $\lambda_1 - K, \lambda_2 - K, \ldots \lambda_n - K$,

are the eigen values of the matrix $(A - KI)$ where K is a non − zero scalar

Proof: Since $\lambda_1, \lambda_2, \ldots . \lambda_n$ are the eigen values of A.

The characteristic polynomial of A is

$|A - \lambda I| = (\lambda_1 - \lambda)(\lambda_2 - \lambda) \ldots (\lambda_n - \lambda)$----------------------1

Thus the characteristic polynomial of A-KI is

$|(A - KI) - \lambda I| = |A - (k+\lambda)I|$

$= [\lambda_1 - (\lambda + K)]\,[\lambda_2 - (\lambda + K)] \ldots . [\lambda_n - (\lambda + K)]$

$= [(\lambda_1 - K) - \lambda][(\lambda_2 - K) - \lambda] \ldots . [(\lambda_n - K) - \lambda]$

Which shows that the eigen values of A-KI are $\lambda_1 - K, \lambda_2 - K, \ldots \ldots . \lambda_n - K$

<u>Theorm 8:</u> If $\lambda_1, \lambda_2 \ldots \lambda_n$ are the eigen values of A find the eigen values of the matrix $(A - \lambda I)^2$

Sol: First we will find the eigen values of the matrix A- λI

Since $\lambda_1, \lambda_2 \ldots \lambda_n$ are the eigen values of A

The characteristics polynomial is

$|A - \lambda I| = (\lambda_1 - K)(\lambda_2 - K) \ldots . (\lambda_n - K)$ ------(1) *where K is scalar*

The characteristic polynomial of the matrix (A- λI) is

$|A - \lambda I - KI| = |A - (\lambda + K)I|$

$= [\lambda_1 - (\lambda + K)]\,[\lambda_2 - (\lambda + K)] \ldots [\lambda_n - (\lambda + K)]$

$= [(\lambda_1 - \lambda) - K)]\,[(\lambda_2 - \lambda) - K] \ldots [(\lambda_n - \lambda) - K)]$

Which shows that eigen values of (A- λI) are $\lambda_1 - \lambda, (\lambda_2 - \lambda) \ldots . \lambda_n - \lambda$

We know that if the eigen values of A are $\lambda_1, \lambda_2 \ldots \lambda_n$ then the eigen values of A^2 are $\lambda_1^2, \lambda_2^2 \ldots . \lambda_n^2$

Theorm 9: If λ is an eigen value of a non-singular matrix A corresponding to the eigen vector then λ^{-1} *is an eigne vector of* A^{-1} *and corresponding eigen vectgor X itself.*

Proof: Since A is non-singular and product of the eigen values is equal to |A|. it follows that none of the eigen vectors of A is o.

If λ is an eigen vector of the non-singular matrix A and X is the corresponding eigen vector $\lambda \neq 0$ and AX= λX. Premultiplying this with A^{-1}, we get $A^{-1}(AX) = A^{-1}(\lambda X)$

$\Rightarrow (A^{-1}A)X = \lambda A^{-1}X \Rightarrow IX = \lambda A^{-1}X$

X = $\lambda A^{-1}X \Rightarrow A^{-1}X = \lambda^{-1}X$

Hence λ^{-1} is an eigen value of A^{-1}

Theorm 10: If λ is an eigen value of a non − singular matrix A, then $\frac{|A|}{\lambda}$ is an eigen value of the matrix Adj A

Proof: Since λ is an eigen value of a non-singular matrix, therfore $\lambda \neq 0$

Also λ is an eigen value of A implies that there exists a non-zero vector X such that AX = λX ------ (1)

$\Rightarrow (adj\ A)AX = (Adj\ A)(\lambda X)$

$\Rightarrow [(adj\ A)A]X = \lambda(\text{adj A})X$

$\Rightarrow |A|IX = \lambda\ (\text{adj A})X$

$\Rightarrow \frac{|A|}{\lambda} X = (adj\ A)X\ on\ (adj\ A)X = \frac{|A|}{\lambda} X$

$\Rightarrow$ *Since X is a non − zero vector, therfore the relation* (1)

it is clear that $\frac{|A|}{X}$ *is an eigen value of the matrix Adj A*

Theorm 11: If λ is an eigen value of an orthogonal matrix then $\frac{1}{\lambda}$ is also an eigen value

Proof: We know that if λ is an eigen value of a matrix A, then $\frac{1}{\lambda}$ is an eigen value of A^{-1}

Since A is an orthogonal matrix, therefore $A^{-1} = A^{1}$

$\frac{1}{\lambda}$ is an eigen value of A^{1}

But the matrices A and A^1 hence the same eigen values, since the determinants |A- Iλ| and |A^1- Iλ| are same.

Hence $\frac{1}{\lambda}$ is also an eigen value of A.

Theorm 12: If λ is eigen value of A then prove that the eigen value of B = $a_0A^2+a_1A+a_2I$ is $a_0\lambda^2+a_1\lambda+a_2$

Proof: If X be the eigen vector corresponding to the eigen value λ, then AX = λX --- (1)

Premultiplying by A on both sides

$\Longrightarrow A(AX) = A(\lambda X)$

$\Longrightarrow A^2X = \lambda(AX) = \lambda(\lambda X) = \lambda^2 X$

This shows that λ^2 *is an eigen vector of* A^2

we have $B = a_0A^2+a_1A+a_2I$

$BX = (a_0A^2+a_1A+a_2I)X$

$= a_0A^2X+a_1AX+a_{2X}$

$= a_0A^2X+a_1\lambda X+a_2X \qquad = (a_0\lambda^2X+a_1\lambda+a_2)X$

$(a_0\lambda^2X+a_1\lambda+a_2)$ is an eigen value of B and the corresponding eigen vector of B is X.

Theorm 14: Suppose that A and P be square matrices of order n such that P is non singular then A and $P^{-1}AP$ have the same eigen values.

Proof: Consider the characteristic equation of $P^{-1}AP$

It is $|(P^{-1}AP)-\lambda I) = |P^{-1}AP-\lambda P^{-1}IP|$

$= |P^{-1}(A-\lambda I)P| = |P^{-1}|\,|A-\lambda I|\,|P|$

$= |A-\lambda I|$ since $|P^{-1}|\,|P| = 1$

Thus the characteristic polynomials of $P^{-1}AP$ and A are same. Hence the eigen values of $P^{-1}AP$ and A are same.

Corollary: If A and B are square matrices such that A is non-singular, then $A^{-1}B$ and BA^{-1} have the same eigen values.

Proof: In the previous theorm take BA^{-1} in place of A and A in place of B.

We deduce that $A^{-1}(BA^{-1})A$ and (BA^{-1}) have the same eigen values.

i.e, $(A^{-1}B)(A^{-1}A)$ and BA^{-1} have the same eigen values.

i.e, $(A^{-1}B)I$ and BA^{-1} have the same eigen values

i.e, $A^{-1}B$ and BA^{-1} have the same eigen values

Corollary2: If A and B are non-singular matrices of the same order, then AB and BA have the same eigen values.

Proof: Notice that $AB=IAB = (B^{-1}B)(AB) = B^{-1}(BA)B$

Using the theorm above BA and $B^{-1}(BA)B$ have the same eigen values.

i.e, BA and AB have the same eigen values.

Theorm 15: The eigen values of a triangular matrix are just the diagonal elements of the matrix.

Proof: Let A = $\begin{vmatrix} a_{11} & a_{12} \dots\dots & a_{1n} \\ 0 & a_{22-\lambda} \dots\dots & a_{2n} \\ \dots\dots & \dots\dots & \dots\dots \\ 0 & 0 \dots\dots & a_{nn} \end{vmatrix}$ be a triangular matrix of order n

The characteristic equation of A is $|A-\lambda I|=0$

i.e., $\begin{vmatrix} a_{11-\lambda} & a_{12} \dots\dots & a_{1n} \\ 0 & a_{22-\lambda} \dots\dots & a_{2n} \\ \dots\dots & \dots\dots & \dots\dots \\ 0 & 0 \dots\dots & a_{nn-\lambda} \end{vmatrix} = 0$

i.e, $(a_{11}-\lambda)(a_{22}-\lambda)\dots\dots(a_{nn}-\lambda)=0$

$\Longrightarrow \lambda = a_{11}, a_{22}, \dots. a_{nn}$

Hence the eigen values of A are $a_{11}, a_{22}, \dots. a_{nn}$ which are just the diagonal elements of A.

Note: lly we can show that the eigen values of a diagonal matrix are just the diagonal elements of the matrix.

<u>Theorm 16</u>: The eigen values of a real symmetric matrix are always real.

Proof: Let λ be an eigen value of a real symmetric matrix A and Let X be the corresponding eigen vector then AX= $\lambda X ----(1)$

Take the conjugate $\bar{A}\bar{X} = \bar{\lambda}\,\bar{X}$

Taking the transpose $\bar{X}^T(\bar{A})^T = \bar{\lambda}\,\bar{X}^T$

Since $\bar{A} = A \text{ and } A^T = A, \text{we have } \bar{X}^T A = \bar{\lambda}\,\bar{X}^T$

Post multiplying by X, we get $\bar{X}^T AX = \bar{\lambda}\,\bar{X}^T X$-------(2)

Premultiplying (1) with $\bar{X}^T$, we get $\bar{X}^T AX = \lambda \bar{X}^T X$ ------ (3)

(1) – (3) gives $(\lambda - \bar{\lambda})\bar{X}^T X = 0$ but $\bar{X}^T X \neq 0 \Rightarrow \lambda - \bar{\lambda} = 0$

$\Rightarrow \lambda - \bar{\lambda} \Rightarrow \lambda$ is real. Hence the result follows

<u>Theorm 17</u>: For a real symmetric matrix, the eigen vectors corresponding to two distinct eigen values are orthogonal.

Proof: Let λ_1, λ_2 be eigen values of a symmetric matrix A and let X_1, X_2 be the corresponding eigen vectors.

Let $\lambda_1 \neq \lambda_2$ we want to show that X_1 is orthogonal to X2 (i.e., $X_1^T X_2 = 0$)

Sice X_1, X_2 are eigen values of A corresponding to the eigen values λ_1, λ_2 we have

$AX_1 = \lambda_1 X_1$ ----- (1) $\quad AX_2 = \lambda_2 X_2$ ------- (2)

Premultiply (1) by X_2^T

$\Longrightarrow X_2^T A X_1 = \lambda_1 X_2^T X_1$

Taking transpose to above, we have

$\Longrightarrow X_1^T A^T (X_2^T)^T = \lambda_1 X_1^T A^T (X_2^T)^T$

$i.e, X_1^T A X_2 = \lambda_1 X_1^T X_2$..............(3)

Premultiplying (2) by $X_1^T, we\ get\ X_1^T A X_2 = \lambda_2 X_1^T X_2 - - - - - (4)$
Hence from (3) and (4) we get
$(\lambda_1 - \lambda_2) X_1^T X_2 = 0$
$\Rightarrow X_1^T X_2 = 0$
$(Q\,\lambda_1 \neq \lambda_2)$
X_1 *is orthogonal to* X_2
Note: If λ is an eigen value of A and f(A) is any polynomial in A, then the eigen value of f(A) is f(λ)

Objective type questions

1. The Eigen values of $\begin{bmatrix} 6 & 3 \\ -2 & 1 \end{bmatrix}$ are [ ]

a) 1,2 b) 2,4 c) 3, 4 d) 1, 5

2. If the determinant of matrix of order 3 is 12. And two eigen values are 1 and 3, then the third eigen value is [ ]

a) 2 b) 3 c) 1 d) 4

3. If A = $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & 4 \\ 0 & 0 & 3 \end{bmatrix}$ then the eigen values of A are [ ]

a) 1, 1, 2 b) 1, 2, 3 c) 1, ½ , 1/3 d) 1, 2, ½

4. The sum of Eigen values of A = $\begin{bmatrix} 1 & -2 & 2 \\ 0 & 1 & 3 \\ 3 & -1 & 2 \end{bmatrix}$ is [ ]

a) 2 b) 3 c) 4 d) 5

5. If the Eigen values of A are (1,-1,2) then the Eogen values of Adj A are [ ]

a) (-2,2,-1) b) (1,1,-2) c) (1,-1,1/2) d) (-1,1,4)

6. If the Eigen values of A are (2,3,4) then the Eigen values of 3A are [ ]

a) 2,3,4 b) ½, 1/3, ¼ c) -2,3,2 d) 6,9,12

7. If the Eigen values of A are (2,3,-2) then the Eigen value of A-3I are [ ]

a) -1,0,-5 b) 2,3,-2 c) -2,-3,2 d) 1,2,2

8. If A is a singular matrix then the product of the Eigen values of A is [ ]

a) 1 b) -1 c) can't be decided d) 0

9. The Eigen vector corresponding to $\times = 2$ □ □ $\begin{bmatrix} 1 & 2 & -1 \\ 0 & 2 & 2 \\ 0 & 0 & -2 \end{bmatrix}$ is [ ]

a) $\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ b) $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ c) $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ d) $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

10. If two Eigen vectors of a symmetric matrix of order 3 are $\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ then the third eigen vector is [ ]

a) $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ b) $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ c) $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ d) $\begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$

11. The Eigen values of $\begin{bmatrix} 5 & 2 \\ -1 & 2 \end{bmatrix}$ are 3 and 4 then the eigen vectors are [ ]

a) $\begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ b) $\begin{bmatrix} 1 \\ -1 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ c) $\begin{bmatrix} 1 \\ -1 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ d) $\begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

12. If the trace of A (2x2 matrix) is 5 and the determinant is 4, then the eigen values are [ ]

a) 2, 2 b) -2, 2 c) -1, -4 d) 1, 4

13. Sum of the eigen values of matrix A is equal to the [ ]

a) Principal diagonal elements of A b) Trace of matrix A c) A&B d) None

14. If A = $\begin{bmatrix} 4 & 2 \\ -3 & 3 \end{bmatrix}$ then A^{-1} = [ ]

a) $\frac{1}{6}$ [7□ − □] b) $\frac{1}{4}$ [5□ − □] c) $\frac{1}{2}$ [7□ − □] d) $\frac{1}{18}$ [7□ − □]

15. If A = $\begin{bmatrix} 6 & 2 \\ 1 & -1 \end{bmatrix}$ then $2A^2-8A-16I$ = [ ]

a) I b) 2A c) A-I d) 5I

16. Similar matrices have same [ ]

a) Characteristic Polynomial b) Characteristic equation

c) Eigen values d) All the above

17. If A = $\begin{bmatrix} 1 & 2 & 5 \\ 0 & -1 & 2 \\ 0 & 0 & 2 \end{bmatrix}$ then A^{-1} = [ ]

a) $\frac{1}{2}$ [□ + □ − □^{2}] b) $\frac{1}{2}$ [□ + □ + □^{2}]

c) $\frac{1}{2}$ [□ + 2□ − □^{2}] d) $\frac{1}{2}$ [□ + 2□ − □^{2}]

18. If A has eigen values (1,2) then the eigen values of $3A+4A^{-1}$ are [ ]

a) 3, 8 b) 7, 11 c) 7, 8 d) 3, 6

19. If A = $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ □h□□ $□^3 =$ [ ]

a) $2A^2+5A$ b) $4A^2+5A$ c) $2A^2+4A$ d) $5A^2+2A$

20. If $D = P^{-1}AP$ then A^2 = [ ]

a) PDP^{-1} b) $P^2D^2(P^{-1})^2$ c) $(P^{-1})^2D^2(p^2)$ d) PD^2P^{-1}

21. The product of Eigen values of A = $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$ □□ [ ]

a) 18 b) -18 c) 36 d) -36

22. If one of the eigen values of A is zero then A is [ ]

a) Singular b) Non-Singular c) Symmetric d) Non-Symmetric

23. If A is a square matrix, D is a diagonal matrix whose elements are eigen values of A and P is the matrix whose Columns are eigen vectors of A^4, then A^4 =________ [ ]

a) PDP^{-1} b) PD^4P^{-1} c) $P^{-1}D^2P$ d) $P^{-1}D^4P$

24. $\frac{|□|}{\times}$ is an eigen value of [ ]

a) Adj A b) A.adj A c) (adj A) A d) None

25. The characteristic equation of $\begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix}$ is [ ]

a) $\times^2 - 3\times +5 = 0$ b) $\times^2 + 3\times +5 = 0$

c) $\times^2 + 3\times -5 = 0$ d) $\times^2 - 3\times -5 = 0$

26. If A = $\begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$ □h□ eigen values of A are 6 and 1 then the model matrix is [ ]

a) $\begin{bmatrix} 4 & 1 \\ 1 & -1 \end{bmatrix}$ b) $\begin{bmatrix} 3 & 1 \\ 1 & -1 \end{bmatrix}$ c) $\begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$ d) $\begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}$

27. If A = $\begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix}$ then the model matrix is [ ]

28. a) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ b) $\begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}$ c) $\begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}$ d) $\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$

29. If A = $\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$ then the model matrix is [ ]

a) $\begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix}$ b) $\begin{bmatrix} 1 & -2 \\ -1 & 1 \end{bmatrix}$ c) $\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$ d) $\begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix}$

30. If A = $\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$ then the spectral matrix is [ ]

a) $\begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix}$ b) $\begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}$ c) $\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$ d) $\begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix}$

30. If A = $\begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}$ then the spectral matrix is [ ]

a) $\begin{bmatrix} -5 & 0 \\ 0 & 5 \end{bmatrix}$ b) $\begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$ c) $\begin{bmatrix} 3 & 0 \\ 0 & -3 \end{bmatrix}$ d) $\begin{bmatrix} 4 & 0 \\ 0 & -4 \end{bmatrix}$

31. If the eigen values of A are 0, 3, 15 then the index and signature of X^TAX are [ ]

a) 2, 1 b) 2,2 c) 3,3 d) 1,1

32. If two eigen vectors of a symmetric matrix are $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ □ □ □ $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ then the third eigen vector is

i. a) $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ b) $\begin{bmatrix} -1 \\ -2 \\ 2 \end{bmatrix}$ c) $\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$ d) $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

[ ]

33. Product of eigen values of matrix A is equal to [ ]

a) determinant of A b) Trace of A c) Principal diagonal of A d) None

34. If A and B are square matrices such that A is non-singular then $A^{-1}B$ and BA^{-1} have [ ]

a) different eigen values b) same eigen values

c) reciprocal eigen values d) None

35. The eigen values of $\begin{bmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$ □ □ □ [ ]

a) 2, 4, 5 b) -2,-4,-5 c) 1,2,3 d) 3,4,6

36. If A = $\begin{bmatrix} 2 & 3 & 5 \\ 0 & -4 & 7 \\ 0 & 0 & 2 \end{bmatrix}$ then A^3-12A = [ ]

a) 12I b) 8I c) 10I d) 16I

37. If A = $\begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$ then $6A^2$-A^3+A = [ ]

a) 5I b) 10I c) 6I d) 8I

38. If A = $\begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}$ then A^3-$4A^2$+A+6I = [ ]

a) [0] b) I c) 3I d) 5I

39. If A = $\begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}$ and $\times = 2\ a$ □ □ 3 then the modal matrix is [ ]

a) $\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$ b) $\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ c) $\begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix}$ d) $\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

40. If A = $\begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$ then D = [ ]

(a) $\begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}$ b) $\begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}$ c) $\begin{bmatrix} 6 & 0 \\ 0 & 1 \end{bmatrix}$ d) $\begin{bmatrix} 6 & 0 \\ 0 & 7 \end{bmatrix}$

41. If A = $\begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}$ then D = [ ]

a) $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ b) $\begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix}$ c) $\begin{bmatrix} 5 & 0 \\ 0 & -2 \end{bmatrix}$ d) $\begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix}$

42. If λ is an eigen value of A then λ^m is eigen value of [ ]

a) A b) A^{-1} c) A^m d) A^{-m}

43. If A = $\begin{bmatrix} -1 & 0 & 0 \\ 2 & -3 & 0 \\ 1 & 4 & 2 \end{bmatrix}$ then the eigen values of A^2 are [ ]

a) -1, -9, -4 b) 1, -3, 2 c) 1, 3, -2 d) 1, 9, 4

44. If λ is the eigen value of A then the eigen values of A^{-1} are [ ]

a) $\frac{|A|}{\lambda}$ b) $\frac{1}{\lambda}$ c) $-\lambda$ d) λ

45. If the eigen values of A are 1, 3, 0 then $|A|$ = [ ]

a) 4 b) 1 c) 3 d) 0

46. The characteristic equation of $\begin{bmatrix} 5 & 2 \\ 3 & 1 \end{bmatrix}$ *is* [ ]

a) $\lambda^2 + 6\lambda + 1 = 0$ b) $\lambda^2 - 6\lambda - 1 = 0$

c) $\lambda^2 + 6\lambda - 1 = 0$ d) $\lambda^2 - 6\lambda + 1 = 0$

47. If A = $\begin{bmatrix} 1 & 2 & -3 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}$ $then\ p^{-1}A^2P =$ [ ]

a) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ b) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}$ c) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{bmatrix}$ d) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}$

48. If A = $\begin{bmatrix} 2 & -2 & 2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$ the eigen values of A are (2, 2, -2) then $p^{-1}A^3P$ = [ ]

a) $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$ b) $\begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & -21 \end{bmatrix}$ c) $\begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$ d) $\begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & -8 \end{bmatrix}$

49. If the eigen values of a matrix are (-2, 3, 6) and the corresponding eigen vectors are $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ then the spectral matrix is [ ]

a) $\begin{bmatrix} -2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix}$ b) $\begin{bmatrix} -1 & 1 & 1 \\ 0 & -1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$

c) $\begin{bmatrix} 4 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 36 \end{bmatrix}$ d) $\begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \\ 0 & -1/\sqrt{3} & 2/\sqrt{6} \\ 1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \end{bmatrix}$

50. If the eigen values of a matrix are (-2, 3, 6) and the corresponding eigen vectors are $\begin{pmatrix}-1\\0\\1\end{pmatrix}, \begin{pmatrix}-1\\-1\\1\end{pmatrix}$ and $\begin{pmatrix}1\\2\\1\end{pmatrix}$ then the spectral matrix is [ ]

a) $\begin{bmatrix}-2 & 0 & 0\\0 & 3 & 0\\0 & 0 & 6\end{bmatrix}$ b) $\begin{bmatrix}-1 & 1 & 1\\0 & -1 & 2\\1 & 1 & 1\end{bmatrix}$

c) $\begin{bmatrix}-1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6}\\0 & -1/\sqrt{3} & 2/\sqrt{6}\\1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6}\end{bmatrix}$ d) $\begin{bmatrix}4 & 0 & 0\\0 & 9 & 0\\0 & 0 & 36\end{bmatrix}$

Unit-II Eigen values and Eigen Vectors [KEY]

1	c	11	d	21	d	31	b	41	b
2	d	12	d	22	a	32	d	42	c
3	b	13	b	23	b	33	a	43	d
4	c	14	d	24	a	34	b	44	b
5	a	15	b	25	a	35	a	45	d
6	d	16	d	26	a	36	d	46	b
7	a	17	c	27	a	37	c	47	c
8	d	18	c	28	a	38	a	48	d
9	a	19	d	29	d	39	a	49	a
10	d	20	d	30	a	40	c	50	b

UNIT - IV

CALCULUS

INTRODUCTION

Let y=f(x) be a function continuous in the closed internal [a,b]. This means that if $a < c < b$,

$\lim_{x\to c} f(x) = f(c)$ and $\lim_{x\to a+0} f(x) = f(a)$, $\lim_{x\to b-0} f(x) = f(b)$

Let y = f(x) be differentiable in the closed interval [a,b]. This means that if $a < c < b$, the derivative of f(x) at x = c exists.

i.e., $\lim_{x\to c} \frac{f(x)-f(c)}{x-c}$ exists

Further $\lim_{x\to a+0} \frac{f(x)-f(a)}{x-a}$ and $\lim_{x\to b-0} \frac{f(x)-f(b)}{x-b}$ exists.

Geometrically, if f(x) in a continuous function in the closed interval [a,b], the graph of y=f(x) is a continuous curve for the points x in [a,b]. If f(x) is derived in closed [a,b], there exists a unique tangent to the curve at every point in the interval [a,b]. This is shown in the following figures (1), (2), & (3).

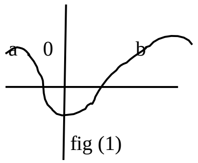


fig (1)

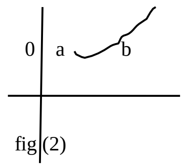
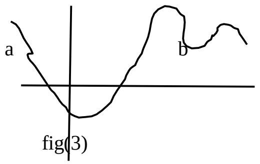


fig (2)



fig(3)

Mean Value Theorems

I) Rolle's Theorem

Statement : Let f(x) be a function such that

i) It is continuous in closed interval [a,b]

ii) It is differentiable in open interval [a,b] and

iii) f(a) = f(b)

Then there exists at least one point c in open interval (a,b) such that $f^1(c)= 0$

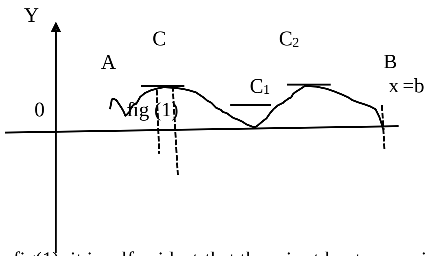
Geometric interpretation of Roll's theorem

Consider the portion AB of the curve y=f(x), lying between x = a and x = b such that

i) It goes continuously from A to B

ii) It has a tangent at every point between A and B, and

iii) Ordinate of A = ordinate of B



From the above fig(1), it is self evident that there is at least one point c (may be more) of the curve at which the tangent is parallel to the x – axis.

i.e. slope of the tangent at c (x = c) = 0. But the slope of the tangent at c is the value of the different co-efficient of f(x) with respect to x, therefore $f^1(c)= 0$.

Hence the theorem is proved.

Eg : 1) Verify Rolle's theorem for the function $f(x) = \frac{\sin x}{ex}$ or $e^{-x} \sin x$ in $[o,\pi]$

Solution : given $f(x) = \frac{\sin x}{ex}$

i) We know that every polynomial is continuous in [a,b] so that sin x & e^{-x} are also continuous function is $[0,\pi]$

$\therefore \frac{\sin x}{ex}$ is also continuous in $[o,\pi]$

ii) Since sin x and e^x are derivable in $[0,\pi]$

$\therefore \frac{\sin x}{ex}$ is also continuous in $[o,\pi]$

iii) $F(o) = \frac{\sin o}{ex} = o$ and $f(\pi) = \frac{\sin}{ex} = 0$

$\therefore f(o) = f(\pi)$

Thus all the three conditions of Roll's theorem are satisfied.

$\therefore$ there exists $c \in (a,b)$ such that $f^1(c) = 0$

$\therefore (c-a)^{m-1} (c – b)^{n-1} [(m+n) c – (mb+na)] = 0$

$\rightarrow (m+n) c – (mb+na) = 0$

$\rightarrow (m+n) c – mb+na$

$\rightarrow c = \frac{mb+na}{m+n} E (a,b)$

[since the point $c \in (a,b)$ divides a and b internally in the ratio m:n]

$\therefore$ Roll's theorem is verified.

(3) verify Rolle's theorem for the function $\log\left[\frac{x2\ +ab}{x(a+b)}\right]$ in [a,b], a > o , b > o

Solution : let $f(x) = \log \frac{x^2-ab}{x(x^2-ab)}$,

$= \log (x^2 +ab) – \log x (a+b)$

$= \log (x^2 +ab) – \log x – \log x(a+b)$

i) Since f(x) is a composite function of continuous functions in [a,b], it is continuous in [a,b].

ii) $f^1(x) = \frac{1}{x^2+ab} 2x - \frac{1}{x} = \frac{x^2-ab}{x(x^2+ab)}$, which exists $\forall$ x∈ (a,b)

∴ f(x) is derivable in (a,b)

iii) $f(a) = \log\left[\frac{a^2+ab}{a^2-ab)}\right] = \log 1 = 0$

$f(b) = \log \left[\frac{b^2+ab}{a^2+ab)}\right] = \log 1 = 0$

∴ f(x) = f(b)

Thus f(x) satisfies all the three conditions of Rolle's theorem.

∴there exists c∈ (a,b) such that $f^1(c) =0$

i.e., $\frac{c^2-ab}{C\,(c^2+ab)} = 0$

i.e., $c^2 – ab = 0$

i.e., $c^2 =ab$

i.e., $c = \pm \sqrt{ab}$

∴ $c = \sqrt{ab} \in (a,b)$

Hence Rolle's theorem is verified.

(4) Using Rolle's theorem, show that $g(x) = gx^3 – 6x^2 – 2x +1$ has a zero between 0 and 1.

Solution:

i) since g(x) being a polynomial.

∴ it is continuous on [0,1]

ii) since the derivative of g(x) exists for all x ε(0,1)

∴ it is derivable on (0,1)

iii) g(0)=1, and g(1) = 8-6-2+1=1

∴ g(0) =g(1)

Hence all the conditions of Rolle's theorem are satisfied on [0,1]

Therefore , there exists a number c ∈ (0,1) such that

$g^1(c) =0$

Now $g^1(x) = 24x^2 - 12x - 2$

$\therefore$ $g^1(c) = 0$

i.e., $24c^2 - 12c - 2 = 0$

i.e., $12c^2 - 6c - 1 = 0$

i.e.,$c = \frac{3 \pm \sqrt{21}}{12}$

i.e. c = 0.63 or -0.132

Here clearly $c = 0.63 \in (0.1)$

Thus there exists at least one root between 0 & 1

5) Verify whether Rolle's therorem can be applied to the following functions in the intervals cited :

i) $f(x) = \tan x$ in $[0,\pi]$

ii) $f(x) = \frac{1}{x^2}$in [-1, 1]

ii) $f(x) = x^3$ in [1,3]

solution:

i) $F(x) = \tan x$ in $[0,\pi]$ since f(x) is discontinuous at $x = \pi/2$

Thus the condition (1) of Rolle's theorem is not satisfied.

Hence we can't apply Rolle's theorem here.

ii) $f(x) = \frac{1}{x^2}$in [-1, 1]

Here f(x) is discontinuous at x = 0

Hence Rolle's theorem can't be applied.

iii) $f(x) = x^3$ in [1,3]

Here clearly f(x) is continuous on [1,3] and derivable on (1,3)

But f(1) G f(3)

i.e., condition (3) of Rolle's theorem fails

Hence we can't apply Rolle's theorem for $f(x) = x^3$ in [1,3]

Exercise : (A)

I) verify Rolle's theorem for the following functions in the intervals indicated.

i) x^2 in [-1,1] ii) $x(x+3)\, e^{-x/2}$ in [-3,0]

iii) $x^{2/3} - 2x^{1/3}$ in (0,8) iv) $\frac{x^2 - x - 6}{x - 1}$ in (-2,3)

v) $x^2 - 2x - 3$ in (1,-3) vi) $|x|$ in [-1,1]

answers : i) c =0 ii) c = -2 iii) c = 1 iv) not applicable

v) c =1 vi) not applicable.

II) Langrange's means value theorem :- (LMVT)

Statement: let f(x) be a function such that

i) It is continuous is closed interval [a,b] and

ii) Differentiable in open interval [a,b]

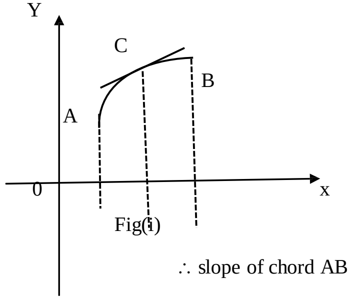
Then there exists at least one point of x say c in open interval (a,b) i.e. a < c < b such that

$$f^1(c) = \frac{f(b)-f(a)}{b-a}$$

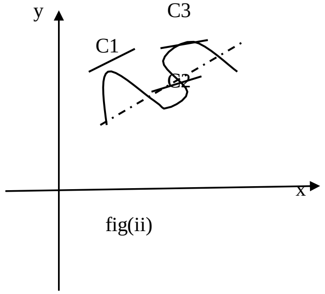
Note : Langrange's mean value theorem is also known as first mean value theorem of differential calculus.

Geometric interpretation of Lagrange's mean value theorem

Let A,B be the points on the curve y = f(x) corresponding to x = a and x=b so that A = [a,f(a)] and B=[b,f(b)], shown in figure (i)&(ii) below.



Fig(i)



fig(ii)

$\therefore$ slope of chord AB = $\frac{f(b)-f(a)}{b-a}$

By lagranges mean value theorem, the slope of the chord AB = $f^1(c)$, the slope of the tangent of the curve at c(x=c)

Hence the lagrange's mean value theorem asserts that if a curve AB has a tangent at each of its points, then there exists at least one point C on this curve, the tangent at which is parallel to the chord AB.

Another form of Lagrange's mean value theorem

Let f(x) be a function such that

i) It is contiunuous in the closed interval [a,a+b],

ii) $f^1(x)$ exists in the open interval (a,a+b)

Then there exists at least one number θ (0 < θ < 1) such that $f(a+b) = f(a) + hf^1(a+\theta b)$

Solved examples

Eg (1) : Verify Lagrange's mean value theorem for

$f(x) = x^3 - x^2 - 5x + 3$ in [0,4]

solution :

Since f(x) is a polynomial so that it is continuous and derivable for every value of x.

In particular, f(x) is continuous in closed interval [0,4] and derivable in open interval (0,4).

Hence by Lagange's mean value theorem, there exists a point c in open interval (0,4) such that

$f^1(c) = \frac{f(4) - f(0)}{4-0}$

i.e., $3c^2 - 2c - 5 = \frac{f(4)-f(0)}{4}$ ------- (1) $(\therefore f^1(x) = 3x^2 - 2x - 5)$

Here $f(4) = 4^3 - 4^2 - 5(4) + 3 = 64 - 16 - 20 + 3 = 31$

and f(0) = 3

from (1), we have $3c^2 - 2c - 5 = 7$

$= 3c^2 - 2c - 12 = 0$

$\therefore c = \frac{2 \pm \sqrt{4+144}}{6} = \frac{2 \pm \sqrt{148}}{6} = \frac{1 \pm \sqrt{37}}{3}$

Here clearly $c = \frac{1 \pm \sqrt{37}}{3}$ c (0,4)

2) Verify lagrage's mean value theorem for $f(x) = \log_e x$ in [1,e]

Solution: given $f(x) = \log_e x$ $\Rightarrow$ $f^1(x) = \frac{1}{x}$

Since f(x) is a polynomial so that it is continuous in [1,e] and derivable in [1,e]

$\therefore$ By lagrage's mean value theorem, there exists a point c c (1,e) such that

$f^1(c) = \frac{f(e) - f(1)}{e-1} = \frac{1-0}{e-1} = \frac{1}{e-1}$ --- (1)

but $f^1(c) = \frac{1}{C}$

$\frac{1}{C} = \frac{1}{e-1}$

$\therefore c = e - 1$ c (1,e)

Hence lagrange's mean value theorem is verified.

3) State whether langrange's mean value theorem can be applied to the following function in the interval indicated justify your answer.

$F(x) = x^{3/4}$ in [-1,1]

Solution :

Given $f(x) = x^{1/3}$

Clearly f(x) is continuous in closed interval[-1,1]

But $f^1(x) = \frac{1}{3}$ x $\frac{1}{3}$-1 = $\frac{1}{3x^{2/3}}$ is not derivable at x = 0.

Hence it is not derivable in open interval (-1,1)

Hence we can't apply lagrange's mean value theorem.

Exercise : (B)

1) Verify lagrange's mean value theorem for the following functions in the intervals indicated.

i) Cos x in [0,π/2] ii) |x| in [-1,1]

iii) x^3-$2x^2$ in [2,5] v) $2x^2 – 7x+10$; a-2, b=5

2) Find C of the lagrange's theorem for

F(x) = (x-1) (x-2) (x-3) on [0,4] ans: C= $\frac{16\pm3}{3}$

3) State whether LMVT can be applicable for the function

F(x) = $\frac{1}{x}$ in [-1,1] ans: not applicable

Eg:

1) If a < b , prove that $\frac{b-a}{1+b^2} < \tan^{-1} a < \frac{b-a}{1+a^2}$ using lagrange's mean value theorem reduce the following

i) $\frac{\pi}{4} + \frac{3}{25} < \tan^{-1}\frac{4}{3} < \frac{\pi}{4} + \frac{1}{6}$

ii) $\frac{5\pi+4}{20} < \tan^{-1}2 < \frac{\pi+2}{4}$

Solution :

Consider $f(x) = \tan^{-1}x$ in [a,b] for 0 < a < b < 1

Since f(x) is continuous in closed interval [a,b] and derivable in open interval[a,b] we can apply lagrange's mean value theorem.

Hence exists a point c c (a,b) such that

$f^1(x) = \frac{f(b)-f(a)}{b-a}$

Hence $f^1(x) = \frac{1}{1+x^2}$

$f^1(c) = \frac{1}{1+c^2}$

Thus there exists a point c, a<c<b such that

$\frac{1}{1+c^2} = \frac{\tan^{-1}b-\tan^{-1}a}{b-a}$ ------- (1)

We have a < c < b

$1+a^2 < 1+c^2 < 1+b^2$ -------- (2)

$\frac{1}{1+a^2} > \frac{1}{1+c^2} > \frac{1}{1+b^2}$

Using 1) and 2), we have

$\frac{1}{1+a^2} > \frac{\tan^{-1} b - \tan^{-1} a}{b-a} > \frac{1}{1+b^2}$

or $\frac{b-a}{1+b^2} < \tan^{-1}b - \tan^{-1}a < \frac{b-a}{1+a^2}$ ------- (3)

Hence the result.

Deduction:

i) We have $\frac{b-}{1+b^2} < \tan^{-1}b - \tan^{-1}a < \frac{b-a}{1+a^2}$ ---- (4)

Put $b = \frac{4}{3}$, a=1, we get

$= \frac{\frac{4}{3}-1}{1+\frac{16}{9}} < \tan^{-1}\left(\frac{4}{3}\right) - \tan^{-1}(1) < \frac{\frac{4}{3}-1}{1+1^2}$

$\rightarrow \frac{\frac{4-3}{3}}{\frac{25}{9}} < \tan^{-1}\left(\frac{4}{3}\right) - \frac{\pi}{4} < \frac{\frac{4-3}{3}}{2}$

$\rightarrow \frac{3}{25} + \frac{\pi}{4} < \tan^{-1}\left(\frac{4}{3}\right) < \frac{\pi}{4} + \frac{1}{6}$

ii) Put b=2 and a=1 in (4), we get

$\frac{2-1}{1+2^2} < \tan^{-1}(2) - \tan^{-1}(1) < \frac{2-1}{1+2^2}$

$\rightarrow \frac{1}{5} < \tan^{-1}(2) - \pi/4 > \frac{1}{2}$

$\rightarrow \frac{1}{5} + \frac{\pi}{4} < \tan^{-1}(2) < \frac{\pi}{4} + \frac{1}{2}$

or $\frac{4+5\pi}{20} < \tan^{-1}(2) < \frac{2+\pi}{4}$

2) Prove that $\frac{\pi}{6} + \frac{1}{5\sqrt{3}} < \sin^{-1}\frac{3}{4} < \frac{\pi}{6} + \frac{1}{8}$ using langrange's mean value theorem.

Solution : let $f(x) = \sin^{-1}(x)$, which is continuous and differentiable .

Now $f^1(x) = \frac{1}{\sqrt{1-x^2}}$ -- $f^1(c) = \frac{1}{\sqrt{1-c^2}}$

By Langrange's mean value theorem, there exist c c (a,b) such that a < c<b and

$f^1(c) = \frac{f(b)-f(a)}{b-a}$

i.e, $\frac{1}{\sqrt{1-c^2}} = \frac{\sin^{-1} b - \sin^{-1} a}{b-a}$ ----- (1)

We have a < c < b

Then $a^2 < c^2 < b^2$

$\rightarrow$ $1-a^2 > 1-c^2 > 1-b^2$

$\rightarrow$ $\sqrt{1-a^2} > \sqrt{1-c^2} > \sqrt{1-b^2}$

$\rightarrow$ $\frac{1}{\sqrt{1-a^2}} > \frac{1}{\sqrt{1-c^2}} > \frac{1}{\sqrt{1-b^2}}$

$\rightarrow$ $\frac{1}{\sqrt{1-a^2}} < \frac{\sin^{-1} b - \sin^{-1} a}{b-a} < \frac{1}{\sqrt{1-b^2}}$

$\rightarrow$ $\frac{b-a}{1+a^2} < \sin^{-1}b - \sin^{-1}a < \frac{b-a}{1+b^2}$

Put a =1/2 and b =3/5

$\rightarrow \frac{\frac{3}{5}-\frac{1}{2}}{\sqrt{1-\frac{1}{4}}} < \sin^{-1}\frac{3}{4} - \sin^{-1}\frac{1}{2} < \frac{\frac{3}{5}-\frac{1}{2}}{\sqrt{1-(\frac{1}{4})^2}}$

$\rightarrow \frac{2}{10\sqrt{3}} < \sin^{-1}\frac{3}{4} - \frac{\pi}{6} < \frac{1}{8}$

$\frac{\pi}{6} + \frac{1}{5\sqrt{3}} < \sin^{-1}\frac{3}{5} < \frac{\pi}{6} + \frac{1}{8}$

3) Prove using mean value theorem $|\sin u - \sin v| \leq |u - v|$

Solution : if u = v , there , is nothing to prove.

If u > v , then consider the function

F(u) = sin u on [v,u]

Clearly, f is continuous on [v,u] and derivable on(v,u)

$\therefore$ By Lagrange's mean valve theorem, there exists c c (v,u)

Such that $\frac{f(u)-f(v)}{u-v} = f^1(c)$

$\frac{\sin u - \sin g\ v}{u-v} = \cos c$

But $|\cos c| \leq 1$

$\therefore$ $\left|\frac{\sin u - \sin g\ v}{u-v}\right| \leq 1$

If v > u, then in similar manner, we have

$|\sin v - \sin u| \leq |v - u|$

$|\sin u - \text{sun } v| \leq |u - v|$ [$\therefore |x| = |x|$]

Hence for all u, v E R

$|\sin u - \text{sun } v| \leq |u - v|$

4) show that for any x> 0 , $1+x < e^x < 1+e^x$

Solution:

Let $f(x) = e^x$ defined on [0,x] and derivable on (0,x)

∴ By Lagrange's mean value theorem

There exists a number c c (0,x) such that

$$\frac{f(x)-f(o)}{x-o} = f^1(c)$$

$$\frac{e^x - e^o}{x} = e^c$$

$$\frac{e^x - 1}{x} = e^c \text{ ------ } (1)$$

Now c ε(o,x) i.e., $0 < c < x$

$$e^o < e^c < e^x$$

$$1 < \frac{e^x - 1}{x} < e^x \; < \text{from } (1) >$$

$$x < e^x - 1 < xe^x$$

$$1+x < e^x < 1+xe^x$$

Exercise : (C)

1) Find c of cauchys mean value theorem for $f(x) = \sqrt{3}$ and $g(x) = \frac{1}{\sqrt{x}}$ in [a,b]

Solutions :

Clearly f, g are continuous on [a,b]

We have $f(x) = \sqrt{x}$

$$F^1(x) = \frac{1}{2\sqrt{x}}$$

And $g(x) = \frac{1}{\sqrt{x}}$

$g^1(x) = -\frac{1}{2x\sqrt{x}}$, which exists on (a,b)

∴ f,g are differentiable on (a,b)

Also $g^1(x) \neq 0 \; \forall \; x \; \varepsilon(a,b) \; CR^+$

∴ conditions of cauchys mean value theorem are satisfied on (a,b)

∴ there exists c c (a,b) such that

$$\frac{f(b)-f(a)}{g(b)-f(a)} = \frac{f^1(c)}{g^1(c)}$$

$$\frac{\sqrt{b}-\sqrt{\;}}{\frac{1}{\sqrt{b}}-\frac{1}{\sqrt{a}}} = \frac{\frac{1}{2\sqrt{c}}}{\frac{1}{2c\sqrt{c}}}$$

$$\frac{\sqrt{b}-\sqrt{a}}{\frac{\sqrt{b}-\sqrt{\;}}{\sqrt{ab}}} = \frac{2c\sqrt{c}}{\sqrt{c}}$$

$$\frac{\sqrt{ab}\,(\sqrt{b}-\sqrt{a})}{\sqrt{b}-\sqrt{a}} = c$$

$$\sqrt{ab} = c$$

Clearly $c = \sqrt{ab}$ c (a,b)

Hence Cauchy mean value theorem is verified.

2) Find c of Cauchy mean value theorem on [a,b] for

$f(x) = e^x$ and $g(x) = e^x$ (a,b > o)

solution :

given (x) = e^x and $g(x) = e^{-x}$

clearly f, g are continuous on[a,b] and f,g are differentiable on (a,b)

also $g^1(x) = - e^{-x} \neq 0 \; \forall$ x ε (a,b) such that

$$\frac{f(b)-f(a)}{g(b)-f(a)} = \frac{f^1(c)}{g^1(c)}$$

$$\frac{e^b - e^a}{e^{-b} - e^{-a}} = \frac{e^c}{-e^{-c}}$$

$$\frac{e^b - e^a}{\frac{1}{e^b} - \frac{1}{e^a}} = -e^{2c}$$

$$\frac{e^b - e^a}{\frac{e^a - e^b}{e^a - e^b}} = - e^{2c}$$

$$\frac{e^b - e^a}{\frac{e^a - e^b}{e^{a+b}}} = - e^{2c}$$

$$\frac{e^{a+b}(e^b - e^a)}{-(e^b - e^a)} = - e^{2c}$$

$e^{a+b} = e^{2c}$

a+b = 2c

C = $\frac{a+b}{2}$ c (a,b)

Hence LMVT is verified

Exercise :(D)

1) Verify cauchy mean value theorem for the following

i) $f(x) = \frac{1}{x^2}$, $g(x) = \frac{1}{x}$ on [a,b] ans: $c = \frac{2ab}{a+b}$

ii) f(x) = sin x , g(x) = cos x on [o, $\frac{\pi}{2}$ ans : c = π/4

iii) f(x) = log x and $g(x) = x^2$ in [a,b], b>a>1 show that $\frac{\log b - \log a}{b-a} = \frac{a+b}{2c2}$

iv) $f(x) = x^2$, $g(x) = x^3$ in [1,2] ans : $c = \frac{14}{9}$

Taylor's theorem

Statement: If f : [a,b] →R is such that

i) f^{n-1} is continuous on [a,b]

ii) $f^{n-1}$is derivable on (a,b) or $f^{(n)}$ exists on (a,b) then there exists a point c c (a,b) such that

$$f(b) = f(a) + \frac{b-a}{1!} f^1(a) + \frac{(b-a)^2}{2!} f^1(a) + \text{---} + \frac{(b-a)^{n-1}}{n-1} f^{n-1}(a) + R_n$$

i) Scholmitch – Roche’s form of remainder:

$$R_n = \frac{(b-a)^p (b-c)^{n-p} f^n(c)}{P(n-1)!} \text{ ------ } (1)$$

ii) Lagrange’s form of remainder : put p=1, in (1) we get

$$R_n = \frac{(b-a)^n f^n(c)}{n!}$$

iii) Cauchy’s form remainder : put p=1 in (1), we get

$$R_n = \frac{(b-a)\ \ (b-c)^{n-1} f^n(c)}{(n-1)!}$$

Note : $(x) = f(a) + (x-a) f^1(a) + \frac{(x-a)^2}{2!} f^{11}(a) + \text{-------}$ is called Taylor’s series for f(x) about x =a

Machlaurin’s theorem

Statement: If f:[0,x] →R is such that

i) f^{n-1} is continuous on [0,x]

ii) f^{n-1} is derivable on(0,x) then there exists a real number θ c (0,1) such that

$$f(x) = f(0) + x f^1(0) + \frac{x^2}{2!} f''(0) + \text{--------} + x^{n-1} f^{(n-1)}(0) + R_n$$

i) **Roche’s form of remainder**:

$$R_n = \frac{x^n (1-\theta)^{n-p} f^n(\theta x)}{p(n-1)!} \text{ --------- } (1)$$

ii) **Langrange’s form remainder** : put p=n in (1)

We get $R_n = \frac{x^n}{n!} f^n(\theta x)$

iii) **Cauchys form of remainder** : put p=1 in (1)

We get $R_n = \frac{x^n (1-\theta)^{n-p} f^n(\theta x)}{(n-1)!}$

Note : $f(x) = f(0) + xf^1(0) + \frac{x^2}{2!} f''''(0) + \text{-----} + \frac{x^n}{n!} f^n(0) + \text{---}$ is called maclaurin’s series expansion of f(x).

Solved examples

1) Obtain Taylor’s series expansion of $f(x) = e^x$ in powers of x+1

Or

Obtain the talylor's series expansion of e^x about x = -1.

Solution : let $f(x) = e^x$ about x = -1

Here a = -1

$\therefore f(x) = e^x \; f^1(x) = e^x \; f^1(a) = e^{-1}$

$f''(x) = e^x$ ----- $f''(a) = e^{-1}$

We know that the Talylor's series expansion of f(x) about x =a is

$f(x) = f(a) + (x-a)\, f^1(a) + \frac{(x-a)^2}{2!} f^{11}(a) +$-------- (1)

put $f(x) = e^x$ & a=-1 in (1) , we get

$e^x = f(-1) + (x+1)\, f^1(-1) + \frac{(x+1)^2}{2!} f^{11}(-1) +$ ------

$e^x = e^{-1} + (x+1)\, e^{-1} + \frac{(x+2)^2}{2!} +$ ------

$e^x = e^{-1}\left[1 + (x+1) + \frac{(x+2)^2}{2!} + \text{------}\right]$ is the required Taylor's series expansion about x =-1

2) Show that $\frac{\sin^{-1}x}{\sqrt{1-x^2}} = x + 4\frac{x^3}{3!} +$ ------

Let $f(x) = \frac{\sin^{-1}x}{\sqrt{1-x^2}}$ then f(0) = 0

$\sqrt{1-x^2}\, f(x) = \sin^{-1}x$ ------ (1)

Differentiating (1) w.r.t. x, we get

$\sqrt{1-x^2}\, f^1(x) + f(x)\left(\frac{-2x}{2\sqrt{1-x^2}}\right) = \frac{1}{\sqrt{1-x^2}}$

$(1-x^2)\, f^1(x) - xf(x) = 1$ -------------(2)

Now $f^1(0) = 1$

Differentiate (2) w.r.t. x, we get

$(1-x^2)\, f^{11}(x) + f^1(x)(-2x) - xf^1(x) - f(x) = 0$ ------

(3) $(1-x^2)\, f^{11}(x) - 3xf^1(x) - f(x) = 0$

Then $f^{11}(0) = 0$

Differentiate (3) w.r.t. x, we get

$(1-x^2)\, f^{111}(x) - 2x\, f^{11}(x) - 3f^1(x) - 3xf^{11}(x) - f^1(x) = 0$

$(1-x^2)\, f^{111}(x) - 5xf^{11}(x) - 4f^1(x) = 0$

$f^{111}(o) - 4f^1(o) = 0$

$f^{111}(o) = 4$ $(\therefore f^1(0) = 1)$

Similarly $f^{IV}(O) = 0$

We have by Taylor's theorem,

$F(x) = f(o) + 1.x + \frac{x^2}{2!} f^{11}(o) + \frac{x^3}{3!} f^{111}(o) + \frac{x^4}{4!} f^{iv}(0) +$----

$\frac{sin^{-1}x}{\sqrt{1-x^2}}$ = 0 +1.x + $\frac{x^2}{2!}$ f^{11}(o) +$\frac{x^3}{3!}$ f^{111}(o) + $\frac{x^4}{4!}$ f^{iv} (0)+----

= x + 4$\frac{x^3}{3!}$ + -----

3) Show that $\log(1+e^x) = \log 2 + \frac{x}{2} + \frac{x^2}{8} - \frac{x^{24}}{192}$ + ----- and hence reduce that

$\frac{e^2}{x+1} = \frac{1}{2} + \frac{x}{4} - \frac{x^b}{48}$ + ------

Solution : let $f(x) = \log(1+e^x)$ then $f(0) = \log 2$

Differentiate successively w.r.t. x, w get

$f^1(x) = \frac{e^x}{1+e^x}$ $\therefore f^1(0) = \frac{1}{1+1} = \frac{1}{2}$

$f^{11}(x) = \frac{(1+e^x)e^x - e^x e^x}{(1+e^x)^2} = \frac{e^x}{(1+e^x)^2}$ $\therefore f^1(o) = \frac{1}{(1+1)^2} = \frac{1}{4}$

$f^{111}(X) = \frac{(1+e^x)^2 e^x - 2e^x(1-e^x)e^x}{(1+e^x)^4} = \frac{(1-e^x)e^x[e^x + e^{2x} - 2e^{2x}]}{(1+e^x)^4}$

$= \frac{e^x - e^{2x}}{(1+e^x)^3}$

$\therefore f^{111}(o) = 0$

$\frac{(1+e^x)^3(e^x-2e^{2x}) - (e^x - e^{2x})\, 3(1+e^x)^2 e^x}{(1+e^x)^6}$

$= \frac{(1+e^x)(e^x-2e^{2x}) - 3e^x(1-1)}{(1-1)^4} = \frac{2}{16} = \frac{1}{8}$

Substituting the values of f(o), f^{II}(o), ---------------in the maclaurin's series

$f(x) = f(0) + xf^1(0) + \frac{x^2}{2!} f^{11}(o) + \frac{x^3}{3!} f^{111}(o)$ + ------

We get $\log(1+e^x) = \log 2 + x(\frac{1}{2}) + \frac{x^2}{2!}(\frac{1}{4}) + \frac{x^3}{3!}(o) + \frac{x^4}{4!}(-\frac{1}{8})$ + ----

$\log(1+e^x) = \log 2 + \frac{x}{2} + \frac{x^2}{8} - \frac{x^4}{192}$ + ----------------(1)

Deduction :

Differentiating the result given by (1) w.r.t x,

We get $\frac{1}{1+e^2} e^x = \frac{1}{2} + \frac{2x}{8} - \frac{x^3}{48}$ + -------------

4) Verify Taylor's theorem for $f(x) = (1-x)^{5/2}$ with lagrange's form of remainder upto 2 terms in the interval [0,1].

Solution: consider $f(x) = (1-x)^{5/2}$ in [0,1]

i) f(x), $f^1(x)$ are continuous in [0,1]

ii) $f^{II}(x)$ is differentiable in (0,1)

Thus f(x) satisfies the conditions of Taylor's theorem.

We consider Taylor's theorem with Lagrange's form of remainder

$f(x) = f(o) + xf^1(o) + \frac{x^2}{2!} f^{11}(o)$ with $0<\theta < 1$ ---- (1)

Here n =p=2, a=0, and x =1

$f(x) = (1-x)^{5/2}$ then $f(0) = 1$

$f^1(x)= \frac{5}{2}(1-x)^{3/2}$ then $f^1(0)= - 5/2$

$f^{II}(x) = \frac{15}{4}(1-x)^{1/2}$ then $f^{II}(\theta x) = \frac{15}{4}(1-\theta x)^{1/2}$

i.e., $f^{II}(\theta) = \frac{15}{4} (1-\theta)^{1/2}$

and $f(1) = 0$

From (1), we have $f(x) = f(0)+xf^1(0) + \frac{x^2}{2!} f^{11}(\theta x)$

Substituting the above values, we get

$\theta = \frac{9}{25}= 0.36$

$\therefore$ θ lies between 0 and 1.

Thus Taylor's theorem is verified.

5) Obtain the Maclaurins series expression of the following functions.

i) e^x ii) sin x iii) $\log_e(1+x)$

solutions:

i) let $f(x) = e^x$ then

$f^1(x) = f^{11}(x) = f^{111}(x) =$ ------- $=e^x$

$\therefore f(0) = f^I(0) = f^{11}(0) = f^{111}(0)$ ------- $=e^0 = 1$

The Maclaurins series expression of f(x) is given by

$f(x) = f(0) + xf^I(0) + \frac{x^2}{2!} f^{11}(0) +$ ------ $+ \frac{x^n}{n!} f^n(0) +$-------

i.e., $e^x = 1+\frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} +$ ------ $+ \frac{x^n}{n!} +$ -----

ii) let $f(x) = \sin x$ then $f(0) = \sin 0 = 0$

Then $f^I(x) = \cos x \rightarrow f^I(0) = \cos 0 = 1$

$f^{11}(x) = - \sin x \rightarrow f^{11}(0) = - \sin 0=0$

$f^{111}(x) = - \cos x \rightarrow f^{111}(0) = - \cos 0=-1$

$f^{IV}(x) = \sin x \rightarrow f^{1v}(0) = \sin 0=0$

substituting all these values in maclarins series of f(x) given by ,

$$f(x) = f(0) + xf^{I}(0) + \frac{x^2}{2!}\ f^{11}(0) + \frac{x^3}{3!}\ f^{111}(0) + \frac{x^4}{4!}\ f^{iv}(0) + \text{-----}$$

$$\sin x = 0 + x(1) + \frac{x^2}{2!}(0) + \frac{x^3}{3!}\ (-1) + \frac{x^4}{4!}(0) + \text{-----}$$

$$\sin x = x - \frac{x^3}{3!} + \text{--------}$$

iii) let $f(x) = \log_e (1+x)$

$$f^{I}(x) = \frac{1}{1+x} \ \rightarrow f^{1}(0) = \frac{1}{1+0} = 1$$

$$f^{II}(x) = \frac{1}{(1+x)^2} \ \rightarrow f^{11}(0) = \frac{1}{(1+0)^2} = 1$$

$$f^{III}(x) = \frac{2}{(1+x)^3} \ \rightarrow f^{111}(0) = \frac{2}{(1+0)^3} = 2$$

$$f^{IV}(x) = \frac{-6}{(1+x)^4} \rightarrow f^{IV}(0) = \frac{-6}{(1+0)^4} = -6$$

substituting all these values in maclurins series expansion of f(x) given by,

$$f(x) = f(0) + xf^{I}(0) + \frac{x^2}{2!} f^{11}(0) + \frac{x^3}{3!}\ f^{111}(0) + \frac{x^4}{4!} f^{iv}(0) + \text{-----}$$

we get , $\log(1-x) = 0+x(1) + \frac{x^2}{2!}\ -1 + \frac{x^3}{3!}\ (2) + \frac{x^4}{4!}\ (-6) + \text{------}$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \text{------}$$

Exercise: (E)

1) Obtain the maclaurins series for the following functions.
   i) Cos x ii) sin x iii) $(1-x)^n$
2) Obtain the Taylor's series expansion of sinx in powers of $x - \frac{\pi}{4}$
3) Write Taylor's series for $f(x) = (1-x)^{5/2}$ with lagrange's form of remainder upto 3 terms in the interval [0,1].

Applications of definite integral's

Definite integral:

Definition

Given a function f(x) that is continuous on the interval [a,b] we divide the interval into n sub intervals of equal width Δx and from each interval choose a point , x_i^*. Then the definite integral of f(x) a to b is

$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$

The integration procedure helps us in evaluating length of plane curves, volume of solids of revolutions, surface area of solids of revolution, determination of centre of mass of a plane mass distribution etc.,

Surface areas of Revolution:

Equation of curve	Axis of revolution	Surface area
Cartesion form: i) Y = f(x)	X – axis	$S = 2\pi\int_a^b y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$
ii) X= f(y)	Y – axis	$S = 2\pi\int_c^d y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dy$

Solved examples

1) Find the area of the surface of the revolution generated by revolving about the x – axis of the arc of the parabola $y^2=12x$ from x =0 to x=3

Solution: given $y^2 = 12x$

$y = 2\sqrt{3}\,\sqrt{x}$

$\frac{dy}{dx} = 2\sqrt{3}\,\frac{1}{2\sqrt{x}} = \sqrt{\frac{3}{x}}$

$\therefore$ Surface area $= 2\pi\int_a^b y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$

$= 2\pi\int_0^3 2\sqrt{3}\,\sqrt{x}\,\sqrt{1+\frac{3}{x}}\,dx$

$= 4\pi\sqrt{3}\int_0^3 \sqrt{x}\,\sqrt{1+\frac{x+3}{x}}\,dx$

$= 4\pi\sqrt{3}\int_0^3 (1+x)^{\frac{1}{2}}\,dx$

$= 4\pi\sqrt{3}\left[\frac{x+3^{3/2}}{3/2}\right]$

$= \frac{8\sqrt{3}}{3}\left[(6)^{3/2} - (3)^{3/2}\right]$

$= \frac{8}{\sqrt{3}}(3)^{3/2}\left[(2)^{3/2} - 1\right]$

$= 24\pi\left[2\sqrt{2} - 1\right]$

2) Find the area of the surface of revolution generates by revolving one area of the curve y=sinx about the x – axis .

Solution: given curve is y = sin x

Here x varies from 0 to π/2

$\therefore \frac{dy}{dx} = \cos x$

Hence required surface area

$$= 2\pi \int_0^{\pi/2} y \sqrt{1 + (\frac{dy}{dx})^2}\, dx$$

$$= 2\pi \int_0^{\pi/2} \sin x \sqrt{1 + cos^2 x}\, dx$$

$$= 2\pi \int_0^{1} \sqrt{1 + t^2}\, dt \quad \text{(putting cos x = t)}$$

$$= 2\pi \left[\frac{t}{2} \sqrt{1 + t^2} + \frac{1}{2} \sinh^{-1} t\right]_0^1$$

$$= 2\pi \ \frac{1}{2} \sqrt{2} + \frac{1}{2} \sin h^{-1} (1) - 0\text{-}0]$$

$$= \pi [\sqrt{2} + \sin h^{-1}(1)]$$

3) The area of the curve $x = y^3$ between y =0 and y=2 is revolved about y-axis. Find the area of surface so generated.

Solution : given curve is $x = y^3$

Then $\frac{dx}{dy} = 3y^2$

$\therefore$ required surface area $= 2\pi \int_0^2 x \sqrt{1 + (\frac{dx}{dy})^2}\, dy$

$$= 2\pi \int_0^2 y^3 \sqrt{1 + (3y^2)^2}\, dy$$

$$= 2\pi \int_0^2 y^3 \sqrt{1 + 9y^4}\, dy$$

$$= 2\pi \int_1^{145} \frac{\sqrt{t}}{36} dt \quad \text{(putting } 1+9y^4=t)$$

$$= \frac{\pi}{18} \left[\frac{2}{3} t^{3/2}\right]_1^{145}$$

$$= \frac{\pi}{27} [(145)^{3/2} - 1]$$

Exercise: (F)

1) Find the surface area generated by the revolution of an arc of the catenary y=C cos h $\frac{x}{c}$ about x –axis ans : $\pi c^2[1+\frac{\sin h2}{2}$

2) Find the area of the surface of revolution generated by revolving the arc of the curve $a^2 y=x^3$ from x =0 to x =a about the x –axis ans: $\frac{\pi}{27}$ $[10\sqrt{10} -1]$

3) Find the surface area of s phere of radius ‘a’ ans: $4\pi a^2$

Volumes of solids of revolution:

Region	Volume of solid generated
Castesion form	
i) y=f(x) the x – axis and the lines x =a , x=b	$V = \pi \int_{a} y^2\, dx$
ii) x=g(y) the y – axis and the lines y=c, y=d	$V = \pi \int$ … dy
iii) y =y1(x), y=y2 (x) the x – axis and ordinates x=a, x=b	$V = \pi \int_a^b (x_2^2 - x_1^2)\, dy$

Solved examples:

1) Find the volume of a sphere of radius 'a'.

Solution :

Sphere is formed by the revolution of the area enclosed by a semi circle its diameter

Equation to circle of radius 'a' is $x^2+y^2 = a^2$ -------(1)

Then $y^2 = a^2-x^2$

In semi circle 'x' varies from –a to a.

$\therefore$ Required volume = $\pi \int_{-a}^{a} y^2\, dx$

$= \pi \int_{-a}^{a} (a^2-x^2)\, dx$

$= \pi \left[a^2x - \frac{x^3}{3}\right]_{-a}^{a}$

$= \pi \left[a^3 - \frac{a^3}{3} + a^3 - \frac{a^3}{3}\right]$

$= \pi \left[2a^2 - \frac{2a^3}{3}\right]$

$= \frac{4\pi a^3}{3}$ cubic units

2) Find the volume of the solid that result when the region enclosed by the curve $y=x^3$, y=0, y=1 is revolved about y – axis .

Solution :

Given curve is $y = x^3$

Then $x=y^{1/3}$

$\therefore$ Required volume = $\pi \int_0^1 x^2\, dy$

$= \pi \int_0^1 (y^{1/3})^2\, dy$

$= \left[\frac{y^{5/3}}{5/3}\right]_0^1$

$= \frac{3\pi}{5} [(1)^{5/3} - 0]$

$= \frac{3\pi}{5}$ cu. units

3) Find the area of the solid generated by revolving the arc of the parabola $x^2 = 12y$, bounded by its latusrectum about y – axis.

Solution:

Given parabola is

$x^2 = 12y = 4(3)y$ (i.e $x^2 = 4ay$)

let 'O' be the vertex and LL^1 be the latusrectum as shown in fig.

for the arc OL, y varies from 0 to 3.

∴ Required volume = 2(volume generated by the revolution about the y – axis of the area OLC)

$= 2\pi \int_0^3 x^2 \, dy$

$= 2\pi \int_0^3 (12)y \, dy$

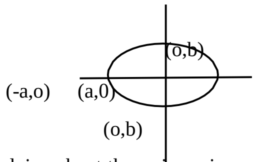
$= 24\pi\left[\frac{y^2}{2}\right]_0^3 = 108\pi$ cubic units

4) Find the volume of the solid generated by revolving the ellipse $\frac{x^2}{a^2} + \frac{y^2}{a^2} = 1$ $(0 < b < a)$ about the major axis.

Solution :

Given equation of the ellipse is

$\frac{x^2}{a^2} + \frac{y^2}{a^2} = 1$



When y = 0, $x = \pm a$

∴ major axis of the ellipse is x = -a to +a

∴ The volume of the solid generated by the given ellipse revolving about the major axis

$= \int_{-a}^{a} \pi y^2 \, dx$

$= 2\pi \int_0^a y^2 \, dx$

$= 2\pi \int_0^a \left(b^2 - \frac{b^2}{a^2} x^2\right) dy$

$= 2\pi \left[b^2 x - \frac{b^2}{a^2} \frac{x^3}{3}\right]_0^a$

$= 2\pi \left[b^2 a - \frac{b^2}{a^2} \frac{a^3}{3} - (0)\right]$

$= 2\pi [ab^2 - \frac{ab^2}{3}] = \frac{4}{3} \pi ab^2$

Exercise :(G)

1) Find the volume got by the revolution of the area bounded by x – axis, the catenary

$y = a \cosh (\frac{x}{a})$ about the x-axis between the ordinates $x = \pm a$

Ans : $\pi a^3 (1+\frac{1}{2}\sinh 2)$

2) Find the volume of the solid when ellipse $\frac{x^2}{a^2} \frac{y^2}{b^2} = 1$, (o< b< a) rotates about minor axis

Ans: $\frac{4\pi a^2 b}{3}$

Beta and gamma functions:-

Definition of improper integral :-

Consider the integral $\int_a^b f(x)$ such an integral for which i) either the interval of integration is not finite i.e a= -œ or b =œ or both ii) or the function f(x) is unbounded at one or more points in [a,b] is called an improper integral.

Eg: $\int_0^\infty \frac{dx}{1+x^4}$, $\int_{-\infty}^\infty \frac{dx}{1+x^2}$, $\int_0^1 \frac{dx}{1-x^2}$ etc..,

Beta function:

The definite integral $\int_0^1 x^{n-1} (1-x)^{n-1}$ dx is called the beta function and is denoted by B(m,n).

i.e., $B(m,n) = \int_0^1 x^{n-1} (1-x)^{n-1} dx$, m>0, n>0

Note : Beta function is also known as Eulerian integral of first kind, which converges for m>0, n>0

Properties of Beta function:

i) Beta function is symmetric i.e. B(m,,n) = B(n,m)

Proof: Since $B(m,n) = \int_0^1 x^{n-1} (1-x)^{n-1} dx$ --------- (1)

We know that $\int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx$ (from properties of definite integral)

$\therefore B(m,n) = \int_0^1 (1-x)^{n-1} [1-(1-x)]^{n-1} dx$

$= \int_0^1 (1-x)^{n-1} x^{n-1} dx$

$= \int_0^1 x^{n-1} (1-x)^{m-1} dx = B(n,m)$ from (1)

$\therefore B(m,n) = B(n,m)$

ii) $B(m,n) = 2 \int_0^{\pi/2} sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$

Proof: We have $B(m,n) = \int_0^1 x^{n-1} (1-x)^{n-1} dx$

Put $x = \sin^2\theta$ so that $dx = \sin 2\theta \, d\theta$

$$\therefore B(m,n) = \int_0^{\pi/2} (\sin^2\theta)^{m-1} (1-\sin^2\theta)^{n-1} \sin 2\theta \, d\theta$$

$$= \int_0^{\pi/2} \sin^{2m-2}\theta \cos^{2n-2}\theta \, (2\sin\theta \cos\theta) \, d\theta$$

$$= 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$$

Or $\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta = \frac{1}{2} B(m,n)$

iii) $B(m,n) = B(m+1,n) + B(m,n+1)$

proof: By definition of Beta function, we have

$$B(m+1,n) + B(m,n+1) = \int_0^1 x^m (1-x)^{n-1} dx + \int_0^1 x^{m-1} (1-x)^n dx$$

$$= \int_0^1 [x^m (1-x)^{n-1} + x^{m-1} (1-x)^n] dx$$

$$= \int_0^1 x^{m-1} (1-x)^{n-1} [x+(1-x)] dx$$

$$= \int_0^1 x^{m-1} (1-x)^{n-1} dx = B(m,n).$$

Hence $B(m,n) = B(m+1,n) + B(m,n+1)$.

Note : If m and n are positive integers, then $B(m,n) = \frac{(m-1)!(n-1)!}{(m+n-1)!}$

Other forms of Beta function:

1) $B(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}} dx$

2) $B(m,n) = \int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}} dx$

3) $B(m,n) = a^m b^n \int_0^\infty \frac{x^{m-1}}{(ax+b)^{m+n}} dx$

4) $\int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}} dx = \frac{B(m,n)}{a^n(1+a)^m}$

5) $\int_a^b (x-b)^{m-1} (a-x)^{n-1} dx = (a-b)^{m+n-1} B(m,n)$, $m > 0$, $n>0$

Solved examples:

1) Express the following integrals in terms of Beta functions

i) $\int_0^1 \frac{x}{\sqrt{1-x^2}} dx$ ii) $\int_0^3 \frac{dx}{\sqrt{9-x^2}}$

Solution:

i) Put $x^2=t$

$x=\sqrt{t}$ so that $dx = \frac{1}{2\sqrt{t}}\, dt$

Limits : If x = 0 , t=0

and x =1,t = 1

$$\therefore \int_0^1 \frac{x}{\sqrt{1-x^2}}\, dx = \int_0^1 \frac{\sqrt{t}}{\sqrt{1-t}} \frac{1}{2\sqrt{t}}\, dt$$

$$= \frac{1}{2} \int_0^1 (1-t)^{-1/2}\, dt$$

$$= \frac{1}{2} \int_0^1 t^{1-1}(1-t)^{1/2-1}\, dt = \frac{1}{2} B(1, \frac{1}{2}) \text{ (by definition of Beta)}$$

ii) Put $x^2 = 9t$

$x = \sqrt{9t} = 3t^{1/2}$

$dx = \frac{3}{2} t^{1/2-1}\, dt$

Limits : When x=0, t=0

x=3, t=1

$$\therefore \int_0^3 \frac{dx}{\sqrt{9-x^2}} = \int_0^1 \frac{1}{\sqrt{9-9t}} \frac{3}{2} t^{1/2-1}\, dt$$

$$= \frac{3}{2} \int_0^1 (9-9t)^{-1/2}\, t^{1/2-1}\, dt$$

$$= \frac{3}{2} \int_0^1 (9)^{-1/2} (1-t)^{-1/2}\, t^{1/2-1}\, dt$$

$$= \frac{3}{2} \frac{1}{3} \int_0^1 t^{\frac{1}{2}-1} (1-t)^{-1/2}\, dt$$

$$= \frac{1}{2} B(\frac{1}{2}, \frac{1}{2})$$

2) Evaluate $\int_0^1 \frac{x^2}{\sqrt{1-x^5}}\, dx$

Solution: consider $\int_0^1 \frac{x^2}{\sqrt{1-25}}\, dx = \int_0^1 x^2 (1-25)^{-1/2}\, dx$

Let $x^5 = t$ so that $x = t^{1/5}$

and $dx = \frac{1}{5} t^{1/5-1}\, dt$

Upper and lower limits are

When x =1, t=1

and x=0,t=0

$$\text{Now} \int_0^1 \frac{x^2}{\sqrt{1-x^5}}\, dx = \int_0^1 x^2 (1-x^5)^{-1/2}\, dx$$

$$= \int_0^1 t^{2/5} (1-t)^{-1/2} \frac{1}{2} t^{1/5-1}\, dt$$

$= \frac{1}{5} \int_0^1 t^{\frac{3}{5}-1} (1-t)^{1/2-1}\, dt$

$= \frac{1}{5} B(\frac{3}{5}, \frac{1}{2})$

Gamma function:

The definite integral $\int_0^\infty e^{-x} x^{n-1}\, dx$, where in>0 is called gamma function and is denoted by (n)

i.e., $(n) = \int_0^\infty e^{-x} x^{n-1}\, dx$

Note : Gamma function is also known as "Eulerian integral of Second kind", which converges only for n>0 and diverges if n≤ 0

Properties of Gamma function:

i) (n)=1 (read as Gamma 1 = 1)

Proof: We have $(n) = \int_0^\infty e^{-x} x^{n-1}\, dx$

$\therefore \quad (1) = \int_0^\infty e^{-x} x^0\, dx$

$= \int_0^\infty e^{-x}\, dx = [-e^{-x}] = -(0-1) = 1$

ii) (n) = (n-1) (n-1) , where n>1

Proof: by definition, we have

$(n) = \int_0^\infty e^{-x} x^{n-1}\, dx = [\, x^{n-1} \frac{e^{-x}}{(-1)}]_0^\infty - \int_0^\infty (n-1)x^{n-2}(\frac{e^{-x}}{-1})\, dx$

(using integration by parts)

$= -\underset{x\to 0}{Lt} \frac{x^{n-1}}{e^x} + 0 + (n-1) \int_0^\infty e^{-x} x^{n-2}\, dx$

$= (n-1) \int_0^\infty e^{-x} x^{n-1}\, dx \qquad (\therefore \ \underset{x\to 0}{Lt} \frac{x^{n-1}}{e^x} = 0 \text{ for } n>1)$

$\therefore$ (n) = (n-1) ⌈(n-1)

Note : 1) (n+1) = n (n)

2) If n is a positive fraction, then we can write

(n) = (n-1) (n-2) ----- (n-r) ⌈(n-r) where (n-r) > 0

3) if n is a non negative integer, then (n+1) = n!

An **important relation between Beta and Gamma functions**:

$B(m,n) = \frac{(m)\ (n)}{(m+n)}$ where m>0, n>0

Proof: from definition, we have $(m) = \int_0^\infty e^{-x} x^{m-1}\, dx$ ---- (1)

Put x=yt so that dx = ydt then (1) gives

$(m) = \int_0^\infty e^{-yt} y^{m-1} t^{m-1}\, ydt = \int_0^\infty y^m\, e^{-yt} t^{m-1}\, dt$

$= \int_0^\infty y^m \, e^{-yx} \, x^{m-1} \, dx$ -----(2)

Or $\frac{(m)}{y^m} = \int_0^\infty e^{-yx} \, x^{m-1} \, dx$ ------------(3)

Multiplying both sides of (3) by $\int_0^\infty e^{-y} y^{m+n-1} dy$, we get

$(m) \int_0^\infty e^{-y} \, y^{n-1} \, dy = \int_0^\infty \int_0^\infty e^{-y(1+x)} y^{m+n-1} x^{m-1} dx \, dy$ --------(4)

Or $(m) \quad (n) = \int_0^\infty \int_0^\infty e^{-y(1+x)} \, y^{m+n-1} \, dy \quad x^{m-1} \, dx$

(by inter changing order of integration)

$\therefore \quad (m) \quad (n) = \int_0^\infty \frac{(m+n)}{(1+x)^{m+n}} \, x^{m-1} \, dx$, by (3)

$= \quad (m+n) \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} \, dx$

$= \quad (m+n) \, B(m,n)$ (from form(1) of Beta function)

Hence $B(m,n) = \frac{(m) \; (n)}{(m+n)}$

Note :

1) $(n) \quad (1-n) = \frac{\pi}{\sin n\pi}$

2) $(n+1) = n \; (n)$ or $(n) = \frac{(n+1)}{n}$ $(n \neq 0, -1, -2, \ldots)$

3) $(1/2) = \sqrt{\pi}$

4) $\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$

5) $\int_{-\infty}^0 e^{-x^2} \, dx = \frac{\sqrt{\pi}}{2}$

6) $\int_{-\infty}^\infty e^{-x^2} dx = \sqrt{\pi}$

7) (n) is defined when n is a negative fraction, But (n) is not defined when n=0 and n is negative integer.

Solved examples:

1) Compute i) $\left(\frac{11}{2}\right)$ ii) $\left(-\frac{1}{2}\right)$

Solution : i) $\left(\frac{11}{2}\right)$

We get that $\Gamma(n) = (n-1)\,(n-2) \ldots (n-r)\,\Gamma(n-r)$ where $(n-r) > 0$

$\therefore \quad \left(\frac{11}{2}\right) = \left(\frac{11}{2} - 1\right) \quad \left(\frac{11}{2} - 1\right)$

$= \frac{9}{2} \quad \left(\frac{9}{2}\right)$

$$= \frac{9}{2} \cdot (\frac{9}{2} - 1) \quad (\frac{9}{2} - 1)$$

$$= \frac{9}{2} \cdot \frac{7}{2} \quad (\frac{7}{2})$$

$$= \frac{9}{2} \cdot \frac{7}{2} \ (\frac{7}{2} - 1) \quad (\frac{7}{2} - 1)$$

$$= \frac{9}{2} \cdot \frac{7}{2} \cdot \frac{5}{2} \quad (\frac{5}{2})$$

$$= \frac{9}{2} \cdot \frac{7}{2} \cdot \frac{5}{2} \cdot (\frac{5}{2} - 1) \quad (\frac{5}{2} - 1)$$

$$= \frac{9}{2} \cdot \frac{7}{2} \cdot \frac{5}{2} \cdot \frac{3}{2} \quad (\frac{3}{2})$$

$$= \frac{9}{2} \cdot \frac{7}{2} \cdot \frac{5}{2} \cdot (\frac{3}{2} - 1) \ \Gamma \ (\frac{3}{2} - 1)$$

$$= \frac{9}{2} \cdot \frac{7}{2} \cdot \frac{5}{2} \cdot \frac{1}{2} \Gamma(1/2)$$

$$= \frac{9}{2} \cdot \frac{7}{2} \cdot \frac{5}{2} \cdot \frac{1}{2} \sqrt{\pi} \qquad (\therefore \ (\frac{1}{2}) = \sqrt{\pi})$$

ii) $(-1/2) = \frac{(-\frac{1}{2}+1)}{-1/2} = -2 \quad (1/2) = 2\sqrt{\pi}$

$(\therefore \ (n) = \frac{\Gamma(n+1)}{n}$) if n is negative fraction)

2) Evaluate

i) $\int_0^2 x\,(8-x^3)^{1/3}\,dx$

ii) $\int_0^{\pi/2} sin^5\theta \ \cos^{7/2}\theta \ d\theta$

iii) $\int_0^{\pi/2} \sqrt{cot\theta} \ d\theta$

Solution :

i) $\int_0^2 x\,(8-x^3)^{1/3}\,dx$

Let $x^3 = 8t$

$x = (8t)^{1/3} = 2t^{1/3}$

$dx = \frac{2}{3}\, t^{1/3-1}\, dt$

when $x=0$; $t=0$

$x = 2$; $t = 1$

$$\therefore \int_0^2 x\,(8-x^3)^{1/3}\,dx = \int_0^2 2t^{1/3}\,(8-8t)^{1/3}\,\frac{2}{3} t^{1/3-1}\,dt$$

$$= \frac{4}{3}\int_0^1 t^{1/3}\,[8(1-t)]^{1/3}\, t^{1/3-1}\,dt$$

$$= \frac{8}{3}\int_0^1 t^{\frac{2}{3}-1}\,(1-t)^{1/3}\,dt$$

$= \frac{8}{3} \int_0^1 t^{\frac{2}{3}-1} (1-t)^{4/3-1}\, dt$

$= \frac{8}{3} B\left(\frac{2}{3}, \frac{4}{3}\right)$ (by definition of Beta function)

$= \frac{8}{3} \cdot \frac{(\frac{2}{3}) \lceil (\frac{4}{3})}{(\frac{2}{3}+\frac{4}{3})}$ ($\therefore$ B(m,n) = $\frac{(m)\ (n)}{(m+n)}$

$= \frac{8}{3} \frac{(\frac{2}{3})(\frac{4}{3}-1) \lceil (\frac{4}{3}-1)}{(2)}$ ($\therefore$ (n) = (n-1) (n-1))

$= \frac{8}{3} \cdot \frac{1}{3} \quad (\frac{2}{3}) \quad (\frac{1}{3})$

$= \frac{8}{9} \quad (\frac{1}{3}) \lceil (1 - \frac{1}{3})$ ($\therefore$ (n) (1-n) = $\frac{\pi}{sin n}$)

$= \frac{8}{9} \frac{\pi}{\sin(\pi/3)} = \frac{16\pi}{9\sqrt{3}}$

ii) solution : put 2m-1 = 5 and 2n-1 = 7/2

so that m=3, n = 9/4

We have $\int_0^{\pi/2} sin^{2m-1}\theta \ \cos^{2n-1}\theta \ d\theta = \frac{1}{2}$ B(m,n)

$\therefore \int_0^{\pi/2} sin^{2m-1}\theta \ \cos^{2n-1}\theta \ d\theta = \frac{1}{2} B\left(3, \frac{9}{4}\right)$

$= \frac{1}{2} \frac{(3)\ (\frac{9}{4})}{(3+\frac{9}{4})}$ ($\therefore$ B(m,n) = $\frac{(m)\ (n)}{(m+n)}$)

$= \frac{1}{2} \frac{(3)\ (\frac{9}{4})}{(\frac{21}{4})}$

$= \frac{(\frac{9}{4})}{(\frac{21}{4})}$

$= \frac{(\frac{9}{4})}{\frac{17}{4} \cdot \frac{13}{4} \cdot \frac{9}{4} \ (\frac{9}{4})}$

$= \frac{64}{1989}$

iii) solution:

$\int_0^{\pi/2} \sqrt{\cot\theta}\ d\theta = \int_0^{\pi/2} \sqrt{\frac{\cos\theta}{sin\theta}}\ d\theta$

$= \int_0^{\pi/2} sin^{-1/2}\theta \ \cos^{1/2}\theta \ d\theta$

Pw 2m-1 = -1/2 and 2n-1 = 1/2

So that m = 1/4, n = 3/4

Then $\int_0^{\pi/2} sin^{-1/2}\theta \ \cos^{1/2}\theta \ d\theta = \frac{1}{2}$ B(m,n)

$= \frac{1}{2} B\left(\frac{1}{4}, \frac{3}{4}\right)$

$$= \frac{1}{2} \quad \frac{(\frac{1}{4})\ (\frac{3}{4})}{(\frac{1}{4}+\frac{3}{4})}$$

$$= \frac{1}{2} \quad (\frac{1}{4}) \quad (\frac{3}{4})$$

$$= \frac{1}{2} \quad (\frac{1}{4}) \quad (1-\frac{1}{4})$$

$$= \frac{1}{2} \frac{\pi}{\sin(\frac{\pi}{4})}$$

$$= \frac{1}{2} \frac{\pi}{(\frac{1}{2})} = \frac{\sqrt{2}\pi}{2} = \frac{\pi}{\sqrt{2}}$$

3) evaluate i) $\int_0^\infty 3^{-4x^2}$ dx ii) $\int_0^1 x^2$ (log1/x)^3 dx

Solution : i) since 3 = $e^{\log 3}$

$\therefore 3^{-4x^2} = e^{-4x^2 log3}$

$\therefore \int_0^\infty 3^{-4x^2}\,dx = \int_0^\infty e^{-4x^2 log3}\ dx$

Put $4x^2$ log3 = t so that $x^2 = \frac{t}{4(log3)}$

$x = \frac{\sqrt{t}}{2\sqrt{log3}}$ ----- (1)

$dx = \frac{1}{2\sqrt{log3}} \frac{1}{2\sqrt{t}}$ dt

When x=0 ; t = 0 (from (1)) x = ∞ ; t = ∞

$\therefore \int_0^\infty 3^{-4x^2}\ dx = \int_0^\infty e^{-4x^2 log3}\ dx$

$= \int_0^\infty e^{-t} \frac{1}{4\sqrt{log3}}\ t^{-1/2}\,dt$

$= \frac{1}{4\sqrt{log3}} \int_0^\infty e^{-t}\ t^{-1/2}\,dt$

$= \frac{1}{4\sqrt{log3}} \int_0^\infty e^{-t}\ t^{1/2-1}\,dt$

$= \frac{1}{4\sqrt{log3}} (\frac{1}{2})$ (by definition of gamma)

$= \frac{1}{4\sqrt{log3}} \sqrt{\pi}$

ii) Put log1/x = t i.e., $\frac{1}{x} = e^t$ or $x = e^{-t}$

$\therefore dx = -e^{-t}\,dt$

When x=1, t=0, t=∞

$\therefore \int_0^1 x^4\ (\log\frac{1}{x})^3\,dx = \int_\infty^0 e^{-4t}\ t^3\ (-e^{-t}\,dt)$

$= \int_0^\infty e^{-5t}\, t^3\, dt$

Pw- 5t = 4 so that dt = $\frac{du}{5}$

$\therefore \int_0^1 x^4 \ (\log \frac{1}{x})^3 \, dx = \int_0^\infty e^{-u} \ (\frac{3}{4})^3 \ \frac{du}{5}$

$= \frac{1}{625} \int_0^\infty e^{-u} \ u^3 \, du$

$= \frac{1}{625} \int_0^\infty e^{-u} \ u^{4-1} \, du$

$= \frac{1}{625}$ (4) ($\therefore$ (n) = $\int_0^\infty e^{-t} \ t^{n-1}\, dt$)

$= \frac{3!}{625} = \frac{6}{625}$

4) prove that $\int_0^\infty \frac{x^8(1-x^6)}{(1-x)^{24}}$ dx = 0 using B-⌠ functions

Solution:

$\int_0^\infty \frac{x^8(1-x^6)}{(1-x)^{24}}\, dx = \int_0^\infty \frac{x^8(1-x^{14})}{(1-x)^{24}}\, dx$

$= \int_0^\infty \frac{x^8 - x^{14})}{(1-x)^{24}}\, dx$

$= \int_0^\infty \frac{x^8}{(1-x)^{24}}\, dx - \int_0^\infty \frac{x^{14}}{(1-x)^{24}}$

$= \int_0^\infty \frac{x^{9-1}}{(1-x)^{9+15}}\, dx - \int_0^\infty \frac{x^{15-1}}{(1-x)^{15+9}}\, dx$

= β(9,15) – β(15,9) ($\therefore$ β(m,n) = $\int_0^\infty \frac{x^{n-1}}{(1-x)^{m+n}}$)

= β (9,15) – β(9,15) ($\therefore$ β(m,n) = B(n,m)

5) Evaluate $\int_0^1 x^3\sqrt{1-x}$ dx using β- ⌠ functions

Solution : $\int_0^1 x^3\sqrt{1-x}\, dx = \int_0^1 x^3\, (1-x)^{1/2}\, dx = \int_0^1 x^{4-1}\, (1-x)^{\frac{3}{2}-1}\, dx$

= β(4. $\frac{3}{2}$) (using defn of Beta function)

$= \frac{(4)\ (\frac{3}{2})}{(4+\frac{3}{2})}$ ($\therefore$ β(m,n) = $\frac{(m)\ (n)}{(m+n)}$

$= \frac{(4)\ (\frac{3}{2})}{(\frac{11}{2})}$ ($\therefore$ (n) = (n-1)!)

$= \frac{3!\ (3/2)}{\frac{9}{2}.\frac{7}{2}.\frac{5}{2}.\frac{3}{2}\ (\frac{3}{2})} = \frac{3!2^4}{9.7.5.3} = \frac{32}{315}$

6) Evaluate $4\int_0^\infty \frac{x^2}{1+x^4}$ dx using β - ⌠ functions.

Solution : put x = $\sqrt{tan\theta}$ so that dx = $\frac{1}{2\sqrt{tan\theta}}$ sec²θ dθ

Also when x = 0, θ=0

And where x -- ∞ , θ--π/2

$\therefore 4\int_0^{\infty} \frac{x^2}{1+x^4} = 4\int_0^{\pi/2} \frac{tan\theta}{1+tan^2\theta} \; \frac{1}{2\sqrt{tan\theta}}\ sec^2\theta\ d\theta$

$= 4\int_0^{\pi/2} \frac{1}{2} \sqrt{tan\theta}\ d\theta$

$= 2\ \int_0^{\pi/2} \frac{tan\theta}{cos\theta} d\theta$

$=2\ \int_0^{\pi/2} sin^{1/2}\ cos^{1/2}\theta\ d\theta$

Put 2n-1 = ½ and 2n-1=-1/2

= m=3/4 and n =1/4

$= 2.\frac{1}{2}\beta(m,n)$ $(\therefore \int_0^{\pi/2} sin^{1/2}\ cos^{1/2}\theta\ d\theta$

$= \beta\left(\frac{3}{4}, \frac{1}{4}\right)$ $= \frac{1}{2}\beta(m,n)$

$= \frac{(\frac{3}{4}) \int}{(\frac{3}{4}+\frac{1}{4})}$ $(\therefore\ \beta(m,n) = \frac{(m)\ \ (n)}{(m+n)})$

$= \ (\frac{1}{4})\ \ (1-\frac{1}{4})$ $(\therefore\ \ (1)=1)$

$= \frac{\pi}{1/2} = \sqrt{2\pi}$

7) Show that $\int_0^1 \frac{x^2}{\sqrt{1-x^4}}\ dx \quad x \int_0^1 \frac{dx}{\sqrt{1-x^2}} \quad = \frac{\pi}{4}$

Solution: let $I_1 = \int_0^1 \frac{x^2}{\sqrt{1-x^4}}\ dx$

Put $x^2 = \ sin\theta$

So that $x^2 = sin^{1/2}\ \theta$

$dx = ½\ sin^{1/2}\theta\ cos\theta\ d\theta$

$\therefore\ \ I_1 = \int_0^1 \frac{x^2}{\sqrt{1-x^4}}\ dx \quad = \int_0^{\pi/2} \frac{sin\theta}{\sqrt{1-sin^2\theta}}\ \frac{1}{2}\ sin^{1/2}\theta\ cos\theta\ d\theta$

$= \frac{1}{2}\int_0^{\pi/2} sin^{1/2}\theta\ \ cos^0\theta\ d\theta$

Put 2m-1 = 1/2 & 2n-1=0

$= \frac{1}{2}.\frac{1}{2}\ \beta(\frac{3}{4}, \frac{1}{2})$ $(\therefore \int_0^{\pi/2} sin^{2m-1}\theta\ \ cos^{2n-1}\ \theta = \frac{1}{2}\beta(m,n)$

$= \frac{1}{4}\ \frac{(\frac{3}{4})\ \ (\frac{1}{2})}{(\frac{3}{4}+\frac{1}{2})}$ $(\therefore \beta(m,n) = \ \frac{(m)\ \ (n)}{(m+n)}$

$= \frac{1}{4}\ \frac{(\frac{3}{4})\ \ \sqrt{\pi}}{(\frac{3}{4})}$ $(\therefore\ \ (\frac{1}{2}) = \sqrt{\pi})$

$$= \frac{\sqrt{\pi}}{4} \frac{(\frac{3}{4})}{(\frac{5}{4}-1)\ (\frac{5}{4}-1)} \qquad (\therefore \quad (n) = (n-1) \quad (n-1)$$

$$\therefore I_1 = \sqrt{t}\ \frac{\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})} \qquad \text{-----------(1)}$$

Now let $I_2 = \int_0^1 \frac{dx}{\sqrt{1-x^4}}$

Put $x^2 = \sin\theta$ so that $x = \sin^{1/2}\theta$

$$dx = \frac{1}{2}\sin^{1/2}\theta \cos\theta\ d\theta$$

$$\therefore I_2 = \int_0^1 \frac{dx}{\sqrt{1-x^4}} \qquad = \int_0^{\pi/2} \frac{1}{2} \frac{\sin_{-1/2}\theta \cos\theta}{\sqrt{1-\sin^2\theta}}$$

$$= \frac{1}{2}\int_0^{\pi/2} \sin^{-1/2}\theta\ d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} \sin^{-1/2}\theta \cos^0\theta\ d\theta$$

$$= \frac{1}{2} \cdot \frac{1}{2}\ \beta(\frac{1}{4}, \frac{1}{\ }) $$

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$$= \frac{1}{4} \frac{(\frac{1}{4})\ (\frac{1}{2})}{(\frac{1}{4}+\frac{1}{2})} \qquad (\therefore \beta(m,n) = \frac{(m)\ (n)}{(m+n)})$$

$$\therefore I_2 = \frac{\sqrt{\pi}\ \Gamma(\frac{1}{4})}{4\ \Gamma(3/4)} \qquad \text{------ (2)}$$

From (1) & (2)

$$I_1 \times I_2 = \int_0^1 \frac{x^2 dx}{\sqrt{1-x^4}} \times \int_0^1 \frac{dx}{\sqrt{1-x^4}} \qquad = \frac{\sqrt{\pi}\ \Gamma(\frac{1}{4})}{4\ \Gamma(1/4)} \times \frac{\sqrt{\pi}}{4} \quad \frac{\Gamma(\frac{1}{4})}{\Gamma(\frac{3}{4})}$$

$$= \frac{\pi}{4}$$

8) Prove that $\int_0^1 \frac{x^2}{\sqrt{1-x^4}}\ dx \times \int_0^1 \frac{dx}{\sqrt{1-x^4}} = \frac{\pi}{4\sqrt{2}}$

Solution : let $I_1 = \int_0^1 \frac{x^2 dx}{\sqrt{1-x^4}}$

Put $x^2 = \sin\theta$ i.e., $x = \sqrt{\sin\theta}$ so that $dx = \frac{1}{2\sqrt{\sin\theta}} \cos\theta\ d\theta$

When $x = 0$, $\theta = 0$

When $x = 1$, $\theta = \pi/2$

$$\therefore I_1 = \int_0^{\pi/2} \frac{\sin\theta}{\sqrt{1-\sin^2\theta}} \frac{\cos\theta\ d\theta}{2\sqrt{\sin\theta}}$$

$$= \frac{1}{2}\int_0^{\pi/2} \sin^{1/2}\theta\ d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} \sin^{1/2}\theta \cos^0\theta\ d\theta$$

$$= \frac{1}{2} \cdot \frac{1}{2}\beta(\frac{3}{4}, \frac{1}{2})$$

$$= \frac{1}{4} \quad \frac{(\frac{1}{4}) \ (\frac{1}{2})}{(\frac{3}{4}+\frac{1}{2})}$$

$$= \frac{1}{4} \quad \frac{(\frac{1}{4}) \ \pi}{(5/4)}$$

$$= \frac{\sqrt{\pi}}{4} \frac{(\frac{3}{4})}{(\frac{5}{4}-1) \ (\frac{5}{4}-1)}$$

$$= \sqrt{\pi} \ \frac{(\frac{3}{4})}{(\frac{1}{4})} \qquad \text{-----------(1)}$$

Let $I_2 = \int_0^1 \frac{dx}{\sqrt{1-x^4}}$

Put $x^2 = \tan\emptyset$ so that $dx = \frac{sec^2 \emptyset}{2\sqrt{tan\theta}} \ d\emptyset$

$$\therefore I_2 = \int_0^{\pi/4} \frac{sec^2\emptyset}{2\sqrt{tan^2\emptyset}\sqrt{tan\emptyset}} \ d\emptyset$$

$$= \frac{1}{2} \int_0^{\pi/4} \frac{sec^2 \emptyset}{sec\emptyset \sqrt{tan\emptyset}} \ d\emptyset$$

$$= \frac{1}{2} \int_0^{\pi/4} \frac{d\emptyset}{\sqrt{sin\emptyset \, cos\emptyset}}$$

$$= \frac{\sqrt{2}}{2} \int_0^{\pi/4} \frac{d\emptyset}{\sqrt{2sin\,\emptyset}}$$

$$= \frac{1}{\sqrt{2}} \int_0^{\pi/2} \frac{cos\emptyset \ d\emptyset}{\sqrt{sin\,2\emptyset}}$$

$$= \frac{1}{\sqrt{2}} \int_0^{\pi/2} \frac{dt}{2\sqrt{sin}} \qquad \text{(putting } 2\emptyset = t)$$

$$= \frac{1}{2\sqrt{2}} \int_0^{\pi/2} sin^{-1/2}t \ dt$$

$$= \frac{1}{2\sqrt{2}} \int_0^{\pi/2} sin^{-1/2}t \ cos^0 t \ dt$$

Put 2m-1 = 1/2 and 2n-1=0

So that m=3/4 and n =1/2

$$= \frac{1}{2\sqrt{2}} \ \frac{1}{2} \ \beta(\frac{1}{4}, \frac{1}{2})$$

$$= \frac{1}{4\sqrt{2}} \ \frac{(\frac{1}{4}) \ (\frac{1}{2})}{(\frac{1}{4}+\frac{1}{2})} = \frac{\sqrt{\pi}}{4\sqrt{2}} \ \frac{(\frac{1}{4})}{(\frac{3}{4})} \qquad \text{-----(2)}$$

$\therefore$ From (1) & (2),

$$I_1 \times I_2 = \int_0^1 \frac{x^2 dx}{\sqrt{1-x^4}} \quad \times \int_0^1 \frac{dx}{\sqrt{1-x^4}}$$

$$= \frac{\sqrt{\pi} \ (\frac{3}{4})}{(\frac{1}{4})} \quad \times \frac{\sqrt{\pi}}{4\sqrt{2}} \frac{(\frac{1}{4})}{(\frac{3}{4})}$$

$$= \frac{\pi}{2\sqrt{2}}$$

9) Prove that $\int_0^1 x^m (\log x)^n \, dx = \frac{(-1)^n n!}{(m+1)^{n+1}}$ where , n, a positive integer and m >-1

Solution :

Put $\log x = -t$ so that $x = e^{-t}$

$dx = -e^{-t} dt$

Also when x = 0, t=∞

x=1 ,t=0

$\therefore \int_0^1 x^m (\log x)^n \, dx = \int_\infty^1 (e^{-t})^m (-t)^n (-e^{-t} dt)$

$= (-1)^n \int_0^\infty e^{-(m+1)t} t^n \, dt$

$= (-1)^n \int_0^\infty e^{-(m+1)t} t^{(n-1)-1} \, dt$

$= (-1)^n \frac{(n+1)}{(m+1)^{n+1}}$ $\quad (\therefore \int_0^\infty e^{-kx} x^{n-1} dx = \frac{(n)}{k^n}$ n> 0 , k> 0)

$= \frac{(-1)^n n!}{(m+1)^{n+1}}$

Note : $\int_0^1 e^m (\log \frac{1}{x})^n dx = \frac{(n+1)}{(m+1)^{n+1}}$

10) Show that

i) $\int_0^\infty x^{n-1} e^{-kx} \, dx = \frac{(n)}{k^n}$ (n> 0 , k>0)

ii) $\int_0^\infty e^{-y^{1/m}} dy = m \quad (m)$

Solution :

i) We know that $(n) = \int_0^\infty x^{n-1} dx$ ------- (1)

Put x = 0, t = 0

x =∞ , t=∞

$\therefore \quad (n) = \int_0^\infty e^{-kt} (kt)^{n-1} (kdt)$ (from (1))

$= k^n \int_0^\infty e^{-kt} t^{n-1} dt$

$= k^n \int_0^\infty e^{-kt} x^{n-1} dx$

Or $\int_0^\infty x^{n-1} e^{-kx} dx = \frac{(n)}{k^n}$

ii) Put $y^{1/m} = x$ i.e.., $y = x^m$ so that $dy = mx^{m-1} dx$

$\therefore \int_0^\infty e^{-y^{1/m}} dy = \int_0^\infty e^{-x} (mx^{m-1}) dx$

$= m \int_0^\infty e^{-x} x^{m-1} dx$

$= m \quad (m)$ (by definition of gamma function)

Exercise :(H)

1) Evaluate $\int_0^a x^4 \sqrt{a^2 - x^2}\, dx$ $\frac{\pi}{32}$

2) Show that $\int_0^1 x^4 (1-\sqrt{x})^5\, dx = 2\beta(10,6)$

3) Evaluate $\int_0^\infty \frac{y^8(1-y^6)}{(1+y)^{24}}\, dy$ ans: 0

4) Show that $\int_0^{\pi/2} \sqrt{sec\theta}\, d\theta = \frac{\Gamma\pi\, \Gamma(\frac{1}{4})}{2\, \Gamma(\frac{3}{4})}$

5) Prove that $\int_0^{\pi/2} \sqrt{cosx}\, dx \times \int_0^{\pi/2} \frac{dx}{\sqrt{cos}} = \pi$

6) Evaluate $\int_0^1 \frac{dx}{\sqrt{-logx}}$ ans : $\sqrt{\pi}$

7) Evaluate $\int_0^{\pi/2} \sqrt{tan\theta} + \sqrt{sec\theta}\, d\theta$ ans: $\frac{1}{2} \quad \Gamma(\frac{1}{4}) + \frac{\sqrt{\pi}}{\Gamma(\frac{3}{4})}]$

8) Prove that $\int_0^\infty \sqrt{x}\, e^{-x^2} dx \times \int_0^\infty e^{-x^2} e^{-x^4} dx$ using β - Γ function and evaluate

9) Show that $\int_0^\infty \sqrt{x}\, e^{-x^2} dx \times \int_0^\infty \frac{e^{-x^2})}{\sqrt{x}} dx = \frac{\pi}{2\sqrt{2}}$

10) Evaluate $\int_0^\infty x^8 e^{-x^2} dx \times \int_0^\infty x^8 e^{-x^4} dx$ ans : $\frac{1}{32} \quad \Gamma(\frac{3}{4}) \quad \Gamma(\frac{3}{8})$

Objective type Questions

1. The value of c of Rolle's theorem for f(x) = $\frac{sinx}{e^x}$ in ((0,Π) is

   a) Π b) $\frac{\pi}{4}$ c) $\frac{\pi}{3}$ d) $\frac{\pi}{2}$

2. Using which mean value theorem, we can calculate approximately the value of $(65)^{1/6}$ in the easier way

   a) Cauch's b) Lagrange's c) Taylor's II order d) Rolle's

3. The value of Cauchy's mean value theorem for (x) = e^x and g(x) = e^{-x} defined on [a,b], o<a<b is

   a) $\sqrt{ab}$ b) $\frac{a-b}{2}$ c) $\frac{a+b}{2}$ d) $\frac{2ab}{a+b}$

4. If f(x) is continuous in [a,b], $f^1(x)$ exists for every value of x in (a,b), f(a)=f(b), there exists at least one value c of x in (a,b) such that $f^1(c)$ = ___________

a) 0   b) a+b   c) c   d) b

5. Lagrange's mean value theorem for f(x) = sec x in (0.2Π) is

   a) Applicable b) not applicable due to non-differentiability

   c) applicable and c=$\frac{\pi}{2}$ d) not applicable due to discontinuity

6. $F(a+h) = f(a) + hf^1(a) + \frac{h^2}{L2} f^{II}(a) + \ldots\ldots\ldots\ldots\ldots + \frac{h^2}{L2} f^n(a+\theta h)$ is called

   a) Taylor's theorem with lagrange form of remainder
   b) Caughy's theorem with lagranges form of remainder
   c) Raiman's theorem with lagrange form of remainder
   d) Lagrange's theorem with lagrange form of remainder

7. If $f(x) = f(o) + \ldots\ldots\ldots F^{II}(o), \frac{x^n}{Ln}$ then the series is called

   a) Maclaurin's Series   b) Taylor's Series

   c) Cauchy's Series   d) lagrange's series

8. The value of Rolle's theorem in (-1,1) for $f(x) = x^3-x$ is

   a) 0   b) $\pm\frac{1}{\sqrt{3}}$ c) $\frac{1}{2}$ d) $\pm\frac{1}{\sqrt{2}}$

9. The value of x so that $\frac{f(b)-f(a)}{b-} f^1(x)$ whne a< x<b given $f(x)= \frac{1}{x^2}$, a=1, b=4

   a) $\frac{3}{4}$ b) $\frac{1}{2}$ c) $\frac{1}{4}$ d) $\frac{9}{4}$

10. The value of c of Cauchy's mean value theorem for the function $f(x) = x^2$, $g(x) = x^3$ in the interval [1,2] is

    a) $\frac{14}{9}$ b) $\frac{3}{14}$ c) $\frac{17}{9}$ d) $\frac{5}{14}$

11. If f(o)=0, $f^1(0)=1$, $f^{II}(0)=1$, $f^{III}(0)=1$, then the machlaurin's expansion of f(x) is given by

    a) $x + \frac{x^2}{2} + \frac{x^3}{3} + \ldots$ b) $x + \frac{x^2}{2} + \frac{x^3}{6} + \ldots$

    c) $-x - \frac{x^2}{2} + \frac{x^3}{3} + \ldots$ d) $x - \frac{x^2}{2} + \frac{x^3}{3} + \ldots$

12. The value c of Rolle's theorem in $[\frac{1}{2}2]$ for $f(x) = x^2 + \frac{1}{x^2}$ is

    a) $\frac{3}{4}$ b) $\frac{5}{4}$ c) 1   d) $\frac{3}{2}$

13. Lagrange's mean value theorem for f(x)=secc in (0,2π) is

    a) Not applicable due to discontinuity   b) applicable & c= $\frac{\pi}{2}$

    c) not applicable due to non differentiable   d) applicable

14. In the Taylor's theorem, the cauchy's form of remainder is

a) $\frac{h^{n-1}f^{n-1}(a-\theta h)}{L n}$ b) $h^n f^n (a+\theta h)$

c) $\frac{h^n(1-\theta)^{n-1}f^n(a-\theta h)}{L n-1}$ d) $\frac{h^{n+1} f^{n} (a-\theta h)}{L n}$

15. The value of c in Rolle's theorem for f(x) = sinx in(0,πa) is

a) $\frac{1}{a}$ b) $\frac{\pi}{4n}$ c) $\frac{\pi}{7n}$ d) $\frac{\pi}{hn}$

16. The value of c in Rolle's theorem for $f(x)=x^2-x$ in (-1,1)

a) 0 b) 0.5 c) 0.25 d) -0.5

17. The value of c in Rolle's theorem for $f(x) = x^2-x(0,1)$

a) 0 b) 0.5 c) 0.25 d) -0.5

18. The value of c in lagrange's mean value theorem for $f(x) = e^x$ in (0,1) is

a) $Log(e-e^{-1})$ b) loge(c) c) log(e+1) d) log (e-1)

19. The value of c in Cauchy's MVT for $f(x)=e^x$ and $g(x) = e^{-x}$ in (3,7) is

a) 4 b) 5 c) 4.5 d) 6

20. The value of θ if $f(x)=x^2$ & $f(x+h)=f(x)+hf^1(x+\theta h)$ a) -0.5 b) 0.25 c) 0 d) 0.5

21. The value of c in Cauchy's mean value theorem for $f(x) =\sqrt{x}$ and $g(x) = \frac{1}{\sqrt{x}}$ in (1,4) is

a) 1.5 b) 2 c) 2.5 d) 3

22. The value of c in lagrange's mean value theorem for f(x) = logx in [1,e] is

a) $(e-1)^{-1}$ b) e+1 c) e-1 d) e

23. Lagrange's mean value theorem is not applicable to the function $f(x) = x^{\frac{1}{3}}$ in [-1,1] because

a) F(-1)≠f(1) b) f is not continuous in [-1,1]

c) f is not derivable in (-1,1) d) f is not a objective function

24. Lagrange's MVT is not applicable to the function defined on [-1,1] by $f(x) =x\sin\frac{1}{x}$ (x≠0) and f(0)=0 because

a) F(-1) =f(1) b) f is not continuous in [-1,1]

c) f is not deriable in (-1,1) d) f is not a one to one function

25. The value of c for lagrange's MVT for the function f(x) =cosx in $[0, \frac{\pi}{2}]$ is

a) $Cos^{-1} (\frac{2}{\pi})$ b) $sin^{-1} (\frac{2}{\pi})$ c) $sin^{-1} (\frac{1}{\pi})$ d) $Cos^{-1} (\frac{1}{\pi})$

26. The value of c for Rolle's theorem for f(x)=(x-a)(x-b) in [a,b] is

a) $-\frac{a+b}{2}$ b) $\sqrt{ab}$ c) a+b d) $\frac{a+b}{2}$

27. The value of c for lagrange's mean value theorem for f(x)=(x-2)(x-3) in [0,1] is a) 0.5 b) 1 c) 2.5 d) 2

28. The value of c of Rolle's theorem for f(x)=(x-1)(x-2) in [0,3] is

a) 1.5 b) 2.5 c) 3 d) 2

29. The value of c of Cauchy's mean value theorem for f(x)=sinx and g(x)=cosx in [0, $\frac{\pi}{2}$]

a) $\frac{\pi}{\delta}$ b) $\frac{\pi}{6}$ c) $\frac{\pi}{4}$ d) $\frac{\pi}{3}$

30. Maclaurin's expansion for log(1+x) is

a) x - $\frac{x^2}{2}$+ $\frac{x^3}{3}$ - $\frac{x^4}{4}$ + …. b) x + $\frac{x^2}{2}$+ $\frac{x^3}{3}$ + $\frac{x^4}{4}$ + ….

b) c) x + $\frac{x^2}{2!}$+ $\frac{x^3}{3!}$ + $\frac{x^4}{4!}$ + …. d) x - $\frac{x^2}{2!}$+ $\frac{x^3}{3!}$ - $\frac{x^4}{4!}$ + ….

31. Maclaurin's expansion of cosx is

a) $\sum_{r=0}^{\infty}\frac{K^{2r}}{(2r)!}$ b) $\sum_{r=0}^{\infty}\frac{(-1)^r K^{2r}}{(2r)!}$

c) $\sum_{r=0}^{\infty}\frac{(-1)^r(K^{2r+1}}{(2r+1)!}$ d) $\sum_{r=0}^{\infty}\frac{K^{2r+1}}{(2r+1)!}$

32. The expansion of e^x in powers of (x-1)

a) E ($\sum_{r=0}^{\infty}\frac{(1-K)^r}{r!}$) b) $e^{-1}\sum_{r=0}^{\infty}\frac{(1-K)^r}{r!}$

c) e($\sum_{r=0}^{\infty}\frac{(-1)^r(K-1)^r}{r!}$) d) $\sum_{r=0}^{\infty}\frac{(-1)^r(K-1)^r}{r!}$

33. The expansion for sinx in powers of (x-$\frac{\pi}{2}$) is

a) 1- $\frac{1}{2}$(x- $\frac{\pi}{2}$)^2 + $\frac{1}{4}$(x - $\frac{\pi}{2}$)^4 - ………………

b) x+(x-$\frac{\pi}{2}$) + $\frac{1}{3!}$(x- $\frac{\pi}{2}$)^3 + ……………….

c) 1+$\frac{1}{2}$(x-$\frac{\pi}{2}$)^2 +$\frac{1}{4!}$(x-$\frac{\pi}{2}$)^4 + ……………..

d) x- (x-$\frac{\pi}{2}$)^2 + $\frac{1}{3!}$(x- $\frac{\pi}{2}$)^3 + ………………

34. Volume of the solid generated by revolving y=f(x), the x-axis and the lines x=a, x=b is

a) $\int_a^b \pi x^2$ dx b) $\int_a^b \pi(y^2 - x^2)$ dx c) $\int_a^b \pi y^2$ dx d) none

35. Volume of the solid generated by revolving the area bounded by the curve x=f(x), the y-axis and the lines y=a, y=b is

a) $\int_a^b \pi x^2$ dx b) $\int_a^b \pi x^2$ dy c) $\int_a^b \pi x^2$ dx d) $\int_a^b \pi y^2$ dy

36. The volume of the sphere of radius 'a' units is

a) $\frac{2\pi a^3}{3}$ b) $\frac{\pi a^3}{3}$ c) πa^3 d) $\frac{4\pi a^3}{3}$

37. The surface area of solid generated yb revolution of circle $x^2+y^2=r^2$ about the diameter is

a) $\frac{2\pi ab^2}{3}$ b) $\frac{4\pi ba^2}{3}$ c) $\frac{4\pi ab^2}{3}$ d) $4\pi ab^2$

b) $r^2 \pi$ b) $2r^2\pi$ c) $3r^2\pi$ d) $4r^2\pi$

38. The surface area of solid generated by revolution of circle $x^2+y^2=r^2$ about the diameter i)

a) $r^2\pi$ b) $2r^2 \pi$ c) $3r^2 \pi$ d) $3r^2\pi$

39. $\int_0^{\pi/2} sin^3x \cos^{5/2} x \, dx$ = ____________

40. $\int_0^{\pi/2} sin^7x \, dx$ = ___________________

41. $\int_0^{\pi/2} tan^{1/2}\theta \, d\theta$ = __________________

42. $\Gamma(3/4)\ \Gamma(1/4)$=__________________

43. $\int_0^{\infty} x^6 e^{-2x} dx$ = __________________

44. $\int_0^1 \frac{xd}{\sqrt{1-x^5}}$ = ________________

45. The value of $\int_0^{\pi/2} sin^p\theta \cos^2\theta \, d\theta$ in terms of β function in _____________

46. The value of $\Gamma(-1/2)$ =______________

47. The value of $\Gamma(1/2)$ = ____________

48. The value of $\Gamma(1)$ = ________________

49. The value of $\beta(\frac{1}{2},\frac{1}{2})$ = _______________

50. The value of $\Gamma(\frac{1}{4})\ \Gamma(\frac{3}{4})$ =_ ___________

51. The value of $\int_0^{\infty} x^{-kx} x^{n-1} dx$ $(n > 0, k > 0)$

52. The value of $\beta(1,2) + \beta(2,1)$ = ____________

53. In terms of β function $\int_0^{\infty} sin^7\theta\sqrt{cos\theta}\, d\theta$ = ___________

54. $\beta(p+1,2) + \beta(p,q+1)$ = ___________________

55. The relation between beta and gamma function is ___________

56. $\int_0^{\infty} e^{-x^2} dx$ = _________________

57. $\int_{-\infty}^{0} e^{-x^2} dx$ = ___________

58. $\int_{-\infty}^{\infty} e^{-x^2} dx$ = ____________

59. $\int_0^{\pi/2} sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$ = _______________

60. If n is a non negative integer, then $\Gamma(n+1)$=___________

UNIT- V

FUNCTION OF SEVERAL VARIABLES

Functions of Several Variable

A Symbol 'Z' which has a definite value for every pair of values of x and y is called a function of two independent variables x and y and we write Z = f(x,y).

Limit of a Function f(x,y):-

The function f(x,y) defined in a Region R, is said to tend to the limit 'ι' as $x\to a$ and $y\to b$ iff corresponding to a positive number $\in$, There exists another positive number δ such that $|f(x,y) - \iota| < \in$ for $0 < (x-a)^2 + (y-b)^2 < \delta^2$ for every point (x,y) in R.

Continuity:-

A function f(x,y) is said to be continuous at the point (a,b) if

$$\lim_{\substack{x\to a\\ y\to b}} f(x,y) = f(a,b).$$

Homogeneous Function:-

An expression of the form,

$a_0 x^n + a_1 x^{n-1} y + a_2 x^{n-2} y^2 + - - - - + a_n y^n$ in which every term is of n^{th} degree, is called a homogeneous function of order 'n'.

Euler's Theorem:-

If z = f(x,y) be a homogeneous function of order 'n' in x and y, then $x\frac{6z}{6x} + Y\frac{6Z}{6F} = nz$

Total Derivatives:-

if u = f(x,y)

where $x = \phi(t)$, $y = \psi(t)$

then $\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$

2) if f(x,y) = c

then

$$\frac{dy}{dx} = -\frac{(\partial u/\partial x)}{(\partial u/\partial y)}$$

3) if u = f(x,y) where $x = \phi(s,t)$, $y = \psi(s,t)$

then

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s}$$

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t}$$

Eulers theroms problems;

1.Verify Eulers therom for the function xy+yz+zx

Sol; Let f(x,y,z)=xy+yz+zx

f(kx,ky,kz)=k^2f(x,y,z)

This is homogeneous fuction of second degree

We have $\frac{6f}{6x}$=y+z $\frac{6f}{6y}$=x+z $\frac{6f}{6z}$=x+y

$x\frac{6f}{6x}+y\frac{6f}{6z}+z\frac{6f}{6z}$ =x(y+z)+y(x+z)+z(x+y)

=xy+xz+yx+yz+zx+zy

=2(xy+yz+zx)

=2f(x,y,z)

PROLEMS:

1.Verify the Eulers therom for z=$\frac{1}{x^2+xy+y^2}$

2.Verify the Eulers therom for u= $\sin^{-1}\frac{x}{y}+\tan^{-1}\frac{y}{x}$

3.Verify the Eulers therom for u= $x^2\tan^{-1}\frac{y}{x}-y^2\tan^{-1}\frac{x}{y}$ **and also prove that** $\frac{6^2u}{6x6y}=\frac{x^2-y^2}{x^2+y^2}$

Jacobian (J) : Let U = u (x , y) , V = v(x , y) are two functions of the independent variables x , y. The jacobian of (u , v) w.r.t (x , y) is given by

$J\left(\frac{u,v}{x,y}\right)=\frac{\partial(u,v)}{\partial(x,y)}=\begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$ Note : $J\left(\frac{u,v}{x,y}\right)\times J\left(\frac{x,y}{u,v}\right)=1$

Similarly of U = u(x, y , z) , V = v (x, y , z) , W = w(x, y , z)

Then the Jacobian of u , v , w w.r.to x , y , z is given by

$$J\left(\frac{u,v,w}{x,y,z}\right)=\frac{\partial(u,v,w)}{\partial(x,y,z)}=\begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix}$$

Solved Problems:

1. **If x + y² = u , y + z² = v , z + x² = w find** $\frac{\partial(x,y,z)}{\partial(u,v,w)}$

Sol : Given $x + y^2 = u$, $y + z^2 = v$, $z + x^2 = w$

We have $\frac{\partial(u,v,w)}{\partial(x,y,z)}=\begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix}=\begin{vmatrix} 1 & 2y & 0 \\ 0 & 1 & 2z \\ 2x & 0 & 1 \end{vmatrix}$

= 1(1-0) – 2y(0 – 4xz) + 0

= 1 – 2y(-4xz)

= 1 + 8xyz

$$\Rightarrow \frac{\partial(x,y,z)}{\partial(u,v,w)}=\frac{1}{\left[\frac{\partial(u,v,w)}{\partial(x,y,z)}\right]}=\frac{1}{1+8xyz}$$

2. S.T the functions $u = x + y + z$, $v = x^2 + y^2 + z^2 - 2xy - 2yz - 2xz$ and $w = x^3 + y^3 + z^3 - 3xyz$ are functionally related. ('07 S-1)

Sol: Given $u = x + y + z$

$v = x^2 + y^2 + z^2 - 2xy - 2yz - 2xz$

$w = x^3 + y^3 + z^3 - 3xyz$

we have

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ 2x-2y-2z & 2y-2x-2z & 2z-2y-2x \\ 3x^2-3yz & 3y^2-3xz & 3z^2-3xy \end{vmatrix}$$

$$=6 \begin{vmatrix} 1 & 1 & 1 \\ x-y-z & y-x-z & z-y-x \\ x^2-yz & y^2-xz & z^2-xy \end{vmatrix}$$

$c_1 \Rightarrow c_1 - c_2$

$c_2 \Rightarrow c_2 - c_3$

$$=6 \begin{vmatrix} 0 & 0 & 1 \\ 2x-2y & 2y-2z & z-y-x \\ x^2-yz-y^2+xz & y^2-xz-z^2+xy & z^2-xy \end{vmatrix}$$

$=6[2(x - y) (y^2 + xy - xz - z^2) - 2(y - z)(x^2 + xz - yz - y^2)]$

$=6[2(x - y)(y - z)(x + y + z) - 2(y - z)(x - y)(x + y + z)]$

$=0$

3. If $x + y + z = u$, $y + z = uv$, $z = uvw$ then evaluate $\frac{\partial(x,y,z)}{\partial(u,v,w)}$ ('06 S-1)

Sol: $x + y + z = u$

$y + z = uv$

$z = uvw$

$y = uv - uvw = uv(1 - w)$

$x = u - uv = u(1 - v)$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{vmatrix}$$

$$= \begin{vmatrix} 1-v & -u & 0 \\ v(1-w) & u(1-w) & -uv \\ vw & uw & uv \end{vmatrix}$$

$R2 \Rightarrow R_2 + R_3$

$$= \begin{vmatrix} 1-v & -u & 0 \\ v & u & 0 \\ vw & uw & uv \end{vmatrix}$$

$= uv[\ u - uv + uv]$

$= u^2v$

4. If $u = x^2 - y^2$, $v = 2xy$ where $x = r\cos\theta$, $y = r\sin\theta$ S.T $\frac{\partial(u,v)}{\partial(r,\theta)} = 4r^3$ ('07 S-2)

Sol: Given $u = x^2 - y^2$ $\quad$ $v = 2xy$

$= r^2\cos^2\theta - r^2\sin^2\theta$ $\quad$ $= 2r\cos\theta\ r\sin\theta$

$= r^2(\cos^2\theta - \sin^2\theta)$ $\quad$ $= r^2\sin 2\theta$

$= r^2\cos 2\theta$

$$\frac{\partial(u,v)}{\partial(r,\theta)} = \begin{vmatrix} u_r & u_\theta \\ v_r & v_\theta \end{vmatrix} = \begin{vmatrix} 2r\cos 2\theta & r^2(-\sin 2\theta)2 \\ 2r\sin 2\theta & r^2(\cos 2\theta)2 \end{vmatrix} = (2r)(2r)\begin{vmatrix} \cos 2\theta & -r\sin 2\theta \\ \sin 2\theta & r(\cos 2\theta) \end{vmatrix}$$

$= 4r^2[r\cos^2 2\theta + r\sin^2 2\theta]$

$= 4r^2(r)[\cos^2 2\theta + \sin^2 2\theta]$

$= 4r^3$

5. If $u = \frac{yz}{x}$, $v = \frac{xz}{y}$, $w = \frac{xy}{z}$ find $\frac{\partial(u,v,w)}{\partial(x,y,z)}$ ('08 S-4)

Sol: Given $u = \frac{yz}{x}$, $v = \frac{xz}{y}$, $w = \frac{xy}{z}$

We have

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix}$$

$u_x = yz(-1/x^2) = \frac{-yz}{x^2}$, $\quad u_y = \frac{z}{x}$, $\quad u_z = \frac{y}{x}$

$v_x = \frac{z}{y}$, $\quad v_y = xz(-1/y^2) = \frac{-xz}{y^2}$, $\quad v_z = \frac{x}{y}$

$w_x = \frac{y}{z}$, $\quad w_y = \frac{x}{z}$, $\quad w_z = xy(-1/z^2) = \frac{-xy}{z^2}$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{-yz}{x^2} & \frac{z}{x} & \frac{y}{x} \\ \frac{z}{y} & \frac{-xz}{y^2} & \frac{x}{y} \\ \frac{y}{z} & \frac{x}{z} & \frac{-xy}{z^2} \end{vmatrix}$$

$$= \frac{1}{x^2}\cdot\frac{1}{y^2}\cdot\frac{1}{z^2}\begin{vmatrix} -yz & xz & xy \\ yz & -xz & xy \\ yz & xz & -xy \end{vmatrix}$$

$$= \frac{(yz)(xz)(xy)}{x^2y^2z^2}\begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

$= 1[-1(1-1) -1(-1-1) + (1+1)]$

$= 0 -1(-2)+(2)$

$=2 + 2$

$=4$

Assignment

Calculate $\frac{\partial(x,y,z)}{\partial(u,v,w)}$ if $x = \sqrt{vw}$, $y = \sqrt{wu}$, $z = \sqrt{uv}$ and $u = r\sin\theta\cos\emptyset$, $v = r\sin\theta\sin\emptyset$, $w = r\cos\theta$

6. If $x = e^r\sec\theta$, $y = e^r\tan\theta$ P.T $\frac{\partial(x,y)}{\partial(r,\theta)} \cdot \frac{\partial(r,\theta)}{\partial(x,y)} = 1$ ('08 S-2)

Sol: Given $x = e^r\sec\theta$, $y = e^r\tan\theta$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} x_r & x_\theta \\ y_r & y_\theta \end{vmatrix}, \quad \frac{\partial(r,\theta)}{\partial(x,y)} = \begin{vmatrix} r_x & r_y \\ \theta_x & \theta_y \end{vmatrix}$$

$x_r = e^r\sec\theta = x$, $x_\theta = e^r\sec\theta\tan\theta$

$y_r = e^r\tan\theta = y$, $y_\theta = e^r\sec^2\theta$

$x^2 - y^2 = e^{2r}(\sec^2\theta - \tan^2\theta)$

⇨ $2r = \log(x^2 - y^2)$

⇨ $r = ½\log(x^2 - y^2)$

$r_x = ½\frac{1}{(x2 - y2)}(2x) = \frac{x}{(x2 - y2)}$

$r_y = ½\frac{1}{(x2 - y2)}(-2y) = \frac{-y}{(x2 - y2)}$

$\frac{x}{y} = \frac{\sec\theta}{\tan\theta} = \frac{1/\cos\theta}{\sin\theta/\cos\theta} = \frac{1}{\sin\theta}$

⇨ $\mathrm{Sin}\theta = \frac{y}{x}$, $\theta = \sin^{-1}(\frac{y}{x})$

$\theta_x = \frac{1}{\sqrt{1-\frac{y^2}{x^2}}}\, y\,(-\frac{1}{x^2}) = \frac{-y}{x\sqrt{x^2-y^2}}$

$\theta_y = \frac{1}{\sqrt{1-\frac{y^2}{x^2}}}(1/x) = \frac{1}{\sqrt{x^2-y^2}}$

$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} e^r\sec\theta\tan\theta \\ e^r\sec2\theta \end{vmatrix} = e^{2r}\sec^2\theta - y\,e^r\sec\theta\tan\theta$

$= e^{2r}\sec\theta[\sec^2\theta - \tan^2\theta] = e^{2r}\sec\theta$

$$\frac{\partial(r,\theta)}{\partial(x,y)} = \begin{vmatrix} \frac{x}{(x2-y2)} & \frac{-y}{(x2-y2)} \\ \frac{-y}{x\sqrt{x^2-y^2}} & \frac{1}{\sqrt{x^2-y^2}} \end{vmatrix}$$

$=[\frac{x}{(x^2-y^2)\sqrt{x^2-y^2}} - \frac{y^2}{x(x^2-y^2)\sqrt{x^2-y^2}}]$

$$= \frac{x^2 - y^2}{x(x^2-y^2)\sqrt{x^2-y^2}} \quad = \frac{1}{x\sqrt{x^2-y^2}} \quad = \frac{1}{e^{2t}\sec\theta}$$

$$\frac{\partial(x,y)}{\partial(r,\theta)} \cdot \frac{\partial(r,\theta)}{\partial(x,y)} = 1$$

Functional Dependence

Two functions u and v are functionally dependent if their Jacobian

$$J\left(\frac{u,v}{x,y}\right) = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = 0$$

If the Jacobian of u, v is not equal to zero then those functions u, v are functionally independent.

** **Maximum & Minimum for function of a single Variable:**

To find the Maxima & Minima of f(x) we use the following procedure.

(i) Find $f'(x)$ and equate it to zero

(ii) solve the above equation we get x_0, x_1 as roots.

(iii) Then find $f^{11}(x)$.

If $f^{11}(x)_{(x = x0)} > 0$, then f(x) is minimum at x_0

If $f^{11}(x)_{(x = x0)}, < 0$, f(x) is maximum at x_0. Similarly we do this for other stationary points.

PROBLEMS:

1. Find the max & min of the function $f(x) = x^5 - 3x^4 + 5$ ('08 S-1)

Sol : Given $f(x) = x^5 - 3x^4 + 5$

$f^1(x) = 5x^4 - 12x^3$

for maxima or minima $f^1(x) = 0$

$5x^4 - 12x^3 = 0$

$X = 0$, $x = 12/5$

$f^{11}(x) = 20x^3 - 36x^2$

At $x = 0 \Rightarrow f^{11}(x) = 0$. So f is neither maximum nor minimum at s = 0

At $x = (12/5)$ $f^{11}(x) = 20(12/5)^3 - 36(12/5)$

$= 144(48-36)/25 = 1728/25 > 0$

So f(x) is minimum at x = 12/5

The minimum value is $f(12/5) = (12/5)^5 - 3(12/5)^4 + 5$

** **Maxima & Minima for functions of two Variables**:

Working procedure:

1. Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ Equate each to zero. Solve these equations for x & y we get the pair of values (a_1,b_1) (a_2,b_2) (a_3,b_3) ………………

2. Find $l = \frac{\partial^2 f}{\partial x^2}$, $m = \frac{\partial^2 f}{\partial x \partial y}$, $n = \frac{\partial^2 f}{\partial y^2}$

i) IF $ln - m^2 > 0$ and $l < 0$ at (a_1,b_1) then f(x ,y) is maximum at (a_1,b_1) and maximum value is $f(a_1,b_1)$.

ii) IF $ln - m^2 > 0$ and $l > 0$ at (a_1,b_1) then f(x ,y) is minimum at (a_1,b_1) and minimum value is $f(a_1,b_1)$.

iii) IF $ln - m^2 < 0$ and at (a_1,b_1) then f(x ,y) is neither maximum nor minimum at (a_1,b_1). In this case (a_1,b_1) is saddle point.

iv) IF $ln - m^2 = 0$ and at (a_1,b_1) , no conclusion can be drawn about maximum or minimum and needs further investigation. Similarly we do this for other stationary points.

PROBLEMS:

1. **Locate the stationary points & examine their nature of the following functions.**

('07 S -2)

$u = x^4 + y^4 - 2x^2 + 4xy - 2y^2$, $(x > 0, y > 0)$

Sol: Given $u(x,y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2$

For maxima & minima $\frac{\partial u}{\partial x} = 0$, $\frac{\partial u}{\partial y} = 0$

$\frac{\partial u}{\partial x} = 4x^3 - 4x + 4y = 0 \Rightarrow x^3 - x + y = 0$ -------------------> (1)

$\frac{\partial u}{\partial y} = 4y^3 + 4x - 4y = 0 \Rightarrow y^3 + x - y = 0$ -------------------> (2)

Adding (1) & (2) ,

$x^3 + y^3 = 0$

$\Rightarrow = x = -y$ -------------------> (3)

(1) $\Rightarrow x^2 - 2x \Rightarrow x = 0, \sqrt{2}, -\sqrt{2}$

Hence (3) $\Rightarrow y = 0, -\sqrt{2}, \sqrt{2}$

$l = \frac{\partial^2 f}{\partial x^2} = 12x^2 - 4$, $m = \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = 4$ & $n = \frac{\partial^2 u}{\partial y^2} = 12y^2 - 4$

$ln - m^2 = (12x^2 - 4)(12y^2 - 4) - 16$

At $(-\sqrt{2}, \sqrt{2})$, $ln - m^2 = (24 - 4)(24 - 4) - 16 = (20)(20) - 16 > 0$

The function has minimum value at $(-\sqrt{2}, \sqrt{2})$

At (0,0) , $ln - m^2 = (0-4)(0-4) - 16 = 0$

(0,0) is not a extrem value.

2. Investigate the maxima & minima if any of the function $f(x) = x^3y^2(1-x-y)$.

('08 S – 4)

Sol: Given $f(x) = x^3y^2(1-x-y) = x^3y^2 - x^4y^2 - x^3y^3$

$\frac{\partial f}{\partial x} = 3x^2y^2 - 4x^3y^2 - 3x^2y^3$ $\frac{\partial f}{\partial y} = 2x^3y - 2x^4y - 3x^3y^2$

For maxima & minima $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$

⇨ $3x^2y^2 - 4x^3y^2 - 3x^2y^3 = 0 \Rightarrow x^2y^2(3 - 4x - 3y) = 0$ ---------------> (1)

⇨ $2x^3y - 2x^4y - 3x^3y^2 = 0 \Rightarrow x^3y(2 - 2x - 3y) = 0$ ----------------> (2)

From (1) & (2) $4x + 3y - 3 = 0$ ----------------X2

$2x + 3y - 2 = 0$ -----------------X3

$2x = 1 \Rightarrow x = ½$

$4(½) + 3y - 3 = 0 \Rightarrow 3y = 3 - 2, y = (1/3)$

$$l = \frac{\partial^2 f}{\partial x^2} = 6xy^2 - 12x^2y^2 - 6xy^3$$

$$\left(\frac{\partial^2 f}{\partial x^2}\right)_{(1/2,1/3)} = 6(1/2)(1/3)^2 - 12(1/2)^2(1/3)^2 - 6(1/2)(1/3)^3 = 1/3 - 1/3 - 1/9 = -1/9$$

$$m = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = 6x^2y - 8x^3y - 9x^2y^2$$

$$\left(\frac{\partial^2 f}{\partial x \partial y}\right)_{(1/2,1/3)} = 6(1/2)^2(1/3) - 8(1/2)^3(1/3) - 9(1/2)^2(1/3)^3 = \frac{6-4-3}{12} = \frac{-1}{12}$$

$$n = \frac{\partial^2 f}{\partial y^2} = 2x^3 - 2x^4 - 6x^3y$$

$$\left(\frac{\partial^2 f}{\partial y^2}\right)_{(1/2,1/3)} = 2(1/2)^3 - 2(1/2)^4 - 6(1/2)^3(1/3) = \frac{1}{4} - \frac{1}{8} - \frac{1}{4} = -\frac{1}{8}$$

$$ln - m^2 = (-1/9)(-1/8) - (-1/12)^2 = \frac{1}{72} - \frac{1}{144} = \frac{2-1}{144} = \frac{1}{144} > 0$$

The function has a maximum value at (1/2 , 1/3)

3. Find three positive numbers whose sum is 100 and whose product is maximum.

('08 S-1)

Sol: Let x ,y ,z be three +ve numbers.

Given x + y + z = 100

⇨ Z = 100 – x – y

Let f (x,y) = xyz =xy(100 – x – y) =100xy $-x^2y-xy^2$

For maxima or minima $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$

$\frac{\partial f}{\partial x} = 100y - 2xy - y^2 = 0$ => y(100- 2x –y) = 0 ----------------> (1)

$\frac{\partial f}{\partial y} = 100x - x^2 - 2xy = 0$ => x(100 –x -2y) = 0 ------------------> (2)

From (1) & (2)

100 -2x –y = 0

200 -2x -4y =0

- - - - - - - - - - - - -

-100 + 3y = 0 => 3y =100 => y =100/3

100 – x –(200/3) = 0 => x = 100/3

$l = \frac{\partial^2 f}{\partial x^2} = -2y$

$\left(\frac{\partial^2 f}{\partial x^2}\right)$ (100/3 , 100/3) = - 200/3

$m = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = 100 - 2x - 2y$

$\left(\frac{\partial^2 f}{\partial x \partial y}\right)$ (100/3 , 100/3) = 100 –(200/3) –(200/3) = -(100/3)

$n = \frac{\partial^2 f}{\partial y^2} = -2x$

$\left(\frac{\partial^2 f}{\partial y^2}\right)$ (100/3 , 100/3) = - 200/3

$ln - m^2 = (-200/3)(-200/3) - (-100/3)^2 = (100)^2/3$

The function has a maximum value at (100/3 , 100/3)

i.e. at x = 100/3, y = 100/3 $\therefore z = 100 - \frac{100}{3} - \frac{100}{3} = \frac{100}{3}$

The required no. are x = 100/3, y = 100/3, z = 100/3

4. Find the maxima & minima of the function $f(x) = 2(x^2 - y^2) - x^4 + y^4$ ('08 S-3)

Sol: Given $f(x) = 2(x^2 - y^2) - x^4 + y^4 = 2x^2 - 2y^2 - x^4 + y^4$

For maxima & minima $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$

$\frac{\partial f}{\partial x} = 4x - 4x^3 = 0$ => $4x(1-x^2) = 0$ => x = 0 , x = ± 1

$\frac{\partial f}{\partial y}$ = -4y + 4y^3 = 0 => -4y (1-y^2) = 0 =>y = 0, y = ± 1

l = $\frac{\partial^2 f}{\partial x^2}$ = 4-12x^2

m = $\frac{\partial^2 f}{\partial x \partial y}$ = $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$ = 0

n = $\frac{\partial^2 f}{\partial y^2}$ = -4 +12y^2

we have ln – m^2 = (4-12x^2)(-4 +12y^2) – 0

= -16 +48x^2 +48y^2 -144x^2y^2

= 48x^2 +48y^2 -144x^2y^2 -16

i) At (0 , ± 1)

ln – m^2 = 0 + 48 - 0 -16 =32 > 0

l = 4-0 = 4 > 0

f has minimum value at (0 , ± 1)

f (x ,y) = 2(x^2 –y^2) –x^4 +y^4

f (0 , ± 1) = 0 – 2 – 0 + 1 = -1

The minimum value is ‘-1 ‘.

ii) At (± 1 ,0)

ln – m^2 = 48 + 0 - 0 -16 =32 > 0

l = 4-12 = - 8 < 0

f has maximum value at (± 1 ,0)

f (x ,y) = 2(x^2 –y^2) –x^4 +y^4

f (± 1 , 0) =2 -0 -1 + 0 = 1

The maximum value is ‘1 ‘.

iii) At (0,0) , (± 1 , ± 1)

ln – m^2 < 0

l = 4 -12x^2

(0 , 0) & (± 1 , ± 1) are saddle points.

F has no max & min values at (0 , 0) , (± 1 , ± 1).

Assignment

1. Find the maximum value of x,y,z when x + y + z = a .

[Ans: $\frac{m^m n^n p^p (a^{m+n+p})}{(m+n+p)^{m+n+p}}$]

***Extremum** : A function which have a maximum or minimum or both is called ‘extremum’

***Extreme value** :- The maximum value or minimum value or both of a function is Extreme value.

***Stationary points**: - To get stationary points we solve the equations $\frac{\partial f}{\partial x}$ = 0 and $\frac{\partial f}{\partial y}$ = 0 i.e the pairs (a_1, b_1), (a_2, b_2) ………….. Are called Stationary.

*Maxima & Minima for a function with constant condition :Lagrangian Method

Suppose f(x , y , z) = 0 ------------(1)

$\emptyset$(x , y , z) = 0 ------------- (2)

F(x , y , z) = f(x , y , z) + γ $\emptyset$(x , y , z) where γ is called Lagrange’s constant.

1. $\frac{\partial F}{\partial x} = 0 \Rightarrow \frac{\partial f}{\partial x} + \gamma \frac{\partial \emptyset}{\partial x} = 0$ --------------- (3)

   $\frac{\partial F}{\partial y} = 0 \Rightarrow \frac{\partial f}{\partial y} + \gamma \frac{\partial \emptyset}{\partial y} = 0$ --------------- (4)

   $\frac{\partial F}{\partial z} = 0 \Rightarrow \frac{\partial f}{\partial z} + \gamma \frac{\partial \emptyset}{\partial z} = 0$ --------------- (5)

2. Solving the equations (2) (3) (4) & (5) we get the stationary point (x, y, z).
3. Substitute the value of x , y , z in equation (1) we get the extremum.

Problem:

1. **Find the minimum value of $x^2 + y^2 + z^2$ given x + y + z =3a (’08 S-2)**

Sol: $u = x^2 + y^2 + z^2$

$\emptyset = x + y + z - 3a = 0$

Using Lagrange’s function

F(x , y , z) = u(x , y , z) + γ $\emptyset$(x , y , z)

For maxima or minima

$\frac{\partial F}{\partial x} = \frac{\partial u}{\partial x} + \gamma \frac{\partial \emptyset}{\partial x} = 2x + \gamma = 0$ ------------ (1)

$\frac{\partial F}{\partial y} = \frac{\partial u}{\partial y} + \gamma \frac{\partial \emptyset}{\partial y} = 2y + \gamma = 0$ ------------ (2)

$\frac{\partial F}{\partial z} = \frac{\partial u}{\partial z} + \gamma \frac{\partial \emptyset}{\partial z} = 2z + \gamma = 0$ ------------ (3)

From (1) , (2) & (3)

$\gamma = -2x = -2y = -2z$

$x = y = z$

$\emptyset = x + x + x - 3a = 0$

$x = a$

$x = y = z = a$

Minimum value of $u = a^2 + a^2 + a^2 = 3a^2$

Fill in the blanks-

1. if u=x+y and v=xy then $\frac{\partial(u,v)}{\partial(x,y)}$ = --------------------------

2. if $x=e^u \cos v$, $y= e^u \sin v$ then $\frac{\partial(x,y)}{\partial(u,v)}$ = ____________________

3. $J\left(\frac{u,v}{u,v}\right)$ =_________________

4. if u= $J\left(\frac{u,v}{x,y}\right)$ then $J\left(\frac{x,y}{u,v}\right)$ = ______________________

5. $\frac{\partial(u,v)}{\partial(x,y)}$ x $\frac{\partial(x,y)}{\partial(u,v)}$ = -------------------------

6. Two functions u and v are said to be functionally dependent if $\frac{\partial(u,v)}{\partial(x,y)}$ = _____________

7. If $u=\frac{x}{y}$ and $v=\frac{x+y}{x-y}$ then $J\left(\frac{u,v}{x,y}\right)$ = _________________

8. If $u=e^x \sin y$, $v=e^x \cos y$ then $\frac{\partial(u,v)}{\partial(x,y)}$ =__________________

9. $J\left(\frac{u,v}{x,y}\right) . J\left(\frac{x,y}{u,v}\right)$= ______________

10. If $x=r\cos\theta$, $y=r\sin\theta$ then $J\left(\frac{x,y}{u,v}\right)$=_________________

11. If u=3x+5y and v=4x-3y then $\frac{\partial(u,v)}{\partial(x,y)}$ =____________

13. If $u=\frac{y}{x}$ and v=xy then $J\left(\frac{u,v}{x,y}\right)$= __________________

OBJECTIVE TYPE QUESTIONS

14. If u=x^2 - 2y ,v=x+y then $\frac{\partial(u,v)}{\partial(x,y)}$ = ________________

(a) $(x+1)^2$ (b) 2(x+1) (c) 3(x+1) (d) None

16. If u(1-v)=x, uv=y then $J\left(\frac{u,v}{x,y}\right) \cdot J\left(\frac{x,y}{u,v}\right)$ =

(a) 0 (b) 1 (c) xy (d) None

17. If u=$\frac{x+y}{1-xy}$, v=tan^{-1} x + tan^{-1} y then $J\left(\frac{u,v}{x,y}\right) \cdot J\left(\frac{x,y}{u,v}\right)$ =

(a) 0 (b) 1 (c) xy (d) None

18. Are u=x$\sqrt{1-x^2}$, v=2x functionally dependent? If so what is $\left(\frac{u,v}{x,y}\right)$?

(a) yes,1 (b) yes,0 (c) No,0 (d) None

19. If u=x^2 y ,v= xy^2 then $\frac{\partial(u,v)}{\partial(x,y)}$ is

(a)5 x^2y^2 (b)4 x^2y^2 (c) $2x^2y^2$ (d) $3x^2y^2$

(Assignment Questions)

{ Functions of Several Variables}

1. If x+y^2=u , y+z^2=v ,z+ x^2=w find $\frac{\partial(x,y,z)}{\partial(u,v,w)}$.
2. If x+y+z=u, y+z= uv, z=uvw then evaluate $\frac{\partial(x,y,z)}{\partial(u,v,w)}$.
3. S.T the functions u=x+y+z, v=$x^2+y^2+z^2$-2xy-2zx and w=$x^3+y^3+z^3$-3xyz are functionally related.
4. Find the max & min values of the function f(x)=x^5-3x^4+5.
5. Find three positive numbers whose sum is 100 and whose product is maximum.
6. Locate the stationary points & examine their nature of the following functions u=$x^4+y^4-2x^2+4xy-2y^2$ (x>0,y>0).
7. If u= $\frac{yz}{x}$,v=$\frac{xz}{y}$, w=$\frac{xy}{z}$, find $\frac{\partial(u,v,w)}{\partial(x,y,z)}$.

UNIT- V

FUNCTION OF SEVERAL VARIABLES

Functions of Several Variable

A Symbol 'Z' which has a definite value for every pair of values of x and y is called a function of two independent variables x and y and we write Z = f(x,y).

Limit of a Function f(x,y):-

The function f(x,y) defined in a Region R, is said to tend to the limit 'ι' as $x \to a$ and $y \to b$ iff corresponding to a positive number ∈, There exists another positive number δ such that $| f(x,y) - \iota | < \in$ for $0 < (x-a)^2 + (y-b)^2 < \delta^2$ for every point (x,y) in R.

Continuity:-

A function f(x,y) is said to be continuous at the point (a,b) if

$$\lim_{\substack{x \to a \\ y \to b}} f(x,y) = f(a,b).$$

Homogeneous Function:-

An expression of the form,

$a_0 x^n + a_1 x^{n-1} y + a_2 x^{n-2} y^2 + - - - - + a_n y^n$ in which every term is of n^{th} degree, is called a homogeneous function of order 'n'.

Euler's Theorem:-

If z = f(x,y) be a homogeneous function of order 'n' in x and y, then $x \frac{6z}{6x} + Y \frac{6Z}{6F} = nz$

Total Derivatives:-

if u = f(x,y)

where x = ϕ(t) , y = ψ(t)

then $\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$

2) if f(x,y) = c

then

$$\frac{dy}{dx} = -\frac{(\partial u/\partial x)}{(\partial u/\partial y)}$$

3) if u = f(x,y) where x = ϕ(s,t), y = ψ(s,t)

then

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s}$$

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t}$$

Eulers theroms problems;

1.Verify Eulers therom for the function xy+yz+zx

Sol; Let f(x,y,z)=xy+yz+zx

f(kx,ky,kz)=k^2f(x,y,z)

This is homogeneous fuction of second degree

We have $\frac{6f}{6x}$=y+z $\frac{6f}{6y}$=x+z $\frac{6f}{6z}$=x+y

$x\frac{6f}{6x}+y\frac{6f}{6z}+z\frac{6f}{6z}$ =x(y+z)+y(x+z)+z(x+y)

=xy+xz+yx+yz+zx+zy

=2(xy+yz+zx)

=2f(x,y,z)

PROLEMS:

1.Verify the Eulers therom for $z=\frac{1}{x^2+xy+y^2}$

2.Verify the Eulers therom for u= $\sin^{-1}\frac{x}{y}+\tan^{-1}\frac{y}{x}$

3.Verify the Eulers therom for u= $x^2\tan^{-1}\frac{y}{x}-y^2\tan^{-1}\frac{x}{y}$ **and also prove that**

$$\frac{6^2u}{6x6y}=\frac{x^2-y^2}{x^2+y^2}$$

Jacobian (J) : Let U = u (x , y) , V = v(x , y) are two functions of the independent variables x , y. The jacobian of (u , v) w.r.t (x , y) is given by

$$J\left(\frac{u,v}{x,y}\right)=\frac{\partial(u,v)}{\partial(x,y)}=\begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$$ Note : $J\left(\frac{u,v}{x,y}\right)\times J\left(\frac{x,y}{u,v}\right)=1$

Similarly of U = u(x, y , z) , V = v (x, y , z) , W = w(x, y , z)

Then the Jacobian of u , v , w w.r.to x , y , z is given by

$$J\left(\frac{u,v,w}{x,y,z}\right)=\frac{\partial(u,v,w)}{\partial(x,y,z)}=\begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix}$$

Solved Problems:

1. **If x + y² = u , y + z² = v , z + x² = w find** $\frac{\partial(x,y,z)}{\partial(u,v,w)}$

Sol : Given $x + y^2 = u$, $y + z^2 = v$, $z + x^2 = w$

We have $$\frac{\partial(u,v,w)}{\partial(x,y,z)}=\begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix}=\begin{vmatrix} 1 & 2y & 0 \\ 0 & 1 & 2z \\ 2x & 0 & 1 \end{vmatrix}$$

= 1(1-0) – 2y(0 – 4xz) + 0

= 1 – 2y(-4xz)

= 1 + 8xyz

$$\Rightarrow \frac{\partial(x,y,z)}{\partial(u,v,w)}=\frac{1}{\left[\frac{\partial(u,v,w)}{\partial(x,y,z)}\right]}=\frac{1}{1+8xyz}$$

2. S.T the functions u = x + y + z , v = $x^2 + y^2 + z^2$ -2xy – 2yz -2xz and w = $x^3 + y^3 + z^3$ -3xyz are functionally related. ('07 S-1)

Sol: Given u = x + y + z

$v = x^2 + y^2 + z^2 - 2xy - 2yz - 2xz$

$w = x^3 + y^3 + z^3 - 3xyz$

we have

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ 2x-2y-2z & 2y-2x-2z & 2z-2y-2x \\ 3x^2-3yz & 3y^2-3xz & 3z^2-3xy \end{vmatrix}$$

$$=6\begin{vmatrix} 1 & 1 & 1 \\ x-y-z & y-x-z & z-y-x \\ x^2-yz & y^2-xz & z^2-xy \end{vmatrix}$$

$c_1 \Rightarrow c_1 - c_2$

$c_2 \Rightarrow c_2 - c_3$

$$=6\begin{vmatrix} 0 & 0 & 1 \\ 2x-2y & 2y-2z & z-y-x \\ x^2-yz-y^2+xz & y^2-xz-z^2+xy & z^2-xy \end{vmatrix}$$

$=6[2(x-y)(y^2 + xy - xz - z^2) - 2(y-z)(x^2 + xz - yz - y^2)]$

$=6[2(x-y)(y-z)(x+y+z) - 2(y-z)(x-y)(x+y+z)]$

$=0$

3. If x + y + z = u , y + z = uv , z = uvw then evaluate $\frac{\partial(x,y,z)}{\partial(u,v,w)}$ ('06 S-1)

Sol: x + y + z = u

y + z = uv

z = uvw

y = uv – uvw = uv(1 – w)

x = u – uv = u (1 – v)

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{vmatrix}$$

$$= \begin{vmatrix} 1-v & -u & 0 \\ v(1-w) & u(1-w) & -uv \\ vw & uw & uv \end{vmatrix}$$

$R2 \Rightarrow R_2 + R_3$

$$= \begin{vmatrix} 1-v & -u & 0 \\ v & u & 0 \\ vw & uw & uv \end{vmatrix}$$

$= uv[\, u - uv + uv]$

$= u^2 v$

4. If $u = x^2 - y^2$, $v = 2xy$ where $x = r\cos\theta$, $y = r\sin\theta$ S.T $\frac{\partial(u,v)}{\partial(r,\theta)} = 4r^3$ ('07 S-2)

Sol: Given $u = x^2 - y^2$, $v = 2xy$

$= r^2\cos^2\theta - r^2\sin^2\theta$ $\qquad = 2r\cos\theta \, r\sin\theta$

$= r^2(\cos^2\theta - \sin^2\theta)$ $\qquad = r^2 \sin 2\theta$

$= r^2 \cos 2\theta$

$$\frac{\partial(u,v)}{\partial(r,\theta)} = \begin{vmatrix} u_r & u_\theta \\ v_r & v_\theta \end{vmatrix} = \begin{vmatrix} 2r\cos 2\theta & r^2(-\sin 2\theta)2 \\ 2r\sin 2\theta & r^2\,(\cos 2\theta)2 \end{vmatrix}$$

$$= (2r)(2r)\begin{vmatrix} \cos 2\theta & -r\sin 2\theta \\ \sin 2\theta & r\,(\cos 2\theta) \end{vmatrix}$$

$$= 4r^2\,[r\cos^2 2\theta + r\sin^2 2\theta\,]$$

$$= 4r^2(r)[\,\cos^2 2\theta + \sin^2 2\theta]$$

$$= 4r^3$$

5. If $u = \frac{yz}{x}$, $v = \frac{xz}{y}$, $w = \frac{xy}{z}$ find $\frac{\partial(u,v,w)}{\partial(x,y,z)}$ ('08 S-4)

Sol: Given $u = \frac{yz}{x}$, $v = \frac{xz}{y}$, $w = \frac{xy}{z}$

We have

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix}$$

$u_x = yz(-1/x^2) = \frac{-yz}{x^2}$, $u_y = \frac{z}{x}$, $u_z = \frac{y}{x}$

$v_x = \frac{z}{y}$, $v_y = xz(-1/y^2) = \frac{-xz}{y^2}$, $v_z = \frac{x}{y}$

$w_x = \frac{y}{z}$, $w_y = \frac{x}{z}$, $w_z = xy\,(-1/z^2) = \frac{-xy}{z^2}$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{-yz}{x^2} & \frac{z}{x} & \frac{y}{x} \\ \frac{z}{y} & \frac{-xz}{y^2} & \frac{x}{y} \\ \frac{y}{z} & \frac{x}{z} & \frac{-xy}{z^2} \end{vmatrix}$$

$$= \frac{1}{x^2}\cdot\frac{1}{y^2}\cdot\frac{1}{z^2}\begin{vmatrix} -yz & xz & xy \\ yz & -xz & xy \\ yz & xz & -xy \end{vmatrix}$$

$$= \frac{(yz)(xz)(xy)}{x^2y^2z^2}\begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

$= 1[-1(1-1) -1(-1-1) + (1+1)]$

$= 0 -1(-2)+(2)$

$=2 + 2$

$=4$

Assignment

Calculate $\frac{\partial(x,y,z)}{\partial(u,v,w)}$ if $x = \sqrt{vw}$, $y = \sqrt{wu}$, $z = \sqrt{uv}$ and $u = r\sin\theta\cos\emptyset$, $v = r\sin\theta\sin\emptyset$, $w = r\cos\theta$

6. If $x = e^r\sec\theta$, $y = e^r\tan\theta$ P.T $\frac{\partial(x,y)}{\partial(r,\theta)} \cdot \frac{\partial(r,\theta)}{\partial(x,y)} = 1$ ('08 S-2)

Sol: Given $x = e^r\sec\theta$, $y = e^r\tan\theta$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} x_r & x_\theta \\ y_r & y_\theta \end{vmatrix}, \quad \frac{\partial(r,\theta)}{\partial(x,y)} = \begin{vmatrix} r_x & r_y \\ \theta_x & \theta_y \end{vmatrix}$$

$x_r = e^r\sec\theta = x$, $\quad x_\theta = e^r\sec\theta\tan\theta$

$y_r = e^r\tan\theta = y$, $\quad y_\theta = e^r\sec^2\theta$

$x^2 - y^2 = e^{2r}(\sec^2\theta - \tan^2\theta)$

⇨ $2r = \log(x^2 - y^2)$

⇨ $r = ½\log(x^2 - y^2)$

$r_x = ½\frac{1}{(x2 - y2)}(2x) = \frac{x}{(x2 - y2)}$

$r_y = ½\frac{1}{(x2 - y2)}(-2y) = \frac{-y}{(x2 - y2)}$

$\frac{x}{y} = \frac{\sec\theta}{\tan\theta} = \frac{1/\cos\theta}{\sin\theta/\cos\theta} = \frac{1}{\sin\theta}$

⇨ $\sin\theta = \frac{y}{x}$, $\theta = \sin^{-1}(\frac{y}{x})$

$$\theta_x = \frac{1}{\sqrt{1-\frac{y^2}{x^2}}}\, y\left(-\frac{1}{x^2}\right) = \frac{-y}{x\sqrt{x^2-y^2}}$$

$$\theta_y = \frac{1}{\sqrt{1-\frac{y^2}{x^2}}}(1/x) = \frac{1}{\sqrt{x^2-y^2}}$$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} e^r\sec\theta & \tan\theta \\ & e^r\sec2\theta \end{vmatrix} = e^{2r}\sec^2\theta - y\, e^r\sec\theta\tan\theta$$

$= e^{2r}\sec\theta[\sec^2\theta - \tan^2\theta] = e^{2r}\sec\theta$

$$\frac{\partial(r,\theta)}{\partial(x,y)} = \begin{vmatrix} \frac{x}{(x2 - y2)} & \frac{-y}{(x2 - y2)} \\ \frac{-y}{x\sqrt{x^2-y^2}} & \frac{1}{\sqrt{x^2-y^2}} \end{vmatrix}$$

$$=\left[\frac{x}{(x^2-y^2)\sqrt{x^2-y^2}} - \frac{y^2}{x(x^2-y^2)\sqrt{x^2-y^2}}\right]$$

$$= \frac{x^2 - y^2}{x(x^2-y^2)\sqrt{x^2-y^2}} \quad = \frac{1}{x\sqrt{x^2-y^2}} \quad = \frac{1}{e^{2r}\sec\theta}$$

$$\frac{\partial(x,y)}{\partial(r,\theta)} \cdot \frac{\partial(r,\theta)}{\partial(x,y)} = 1$$

Functional Dependence

Two functions u and v are functionally dependent if their Jacobian

$$J\left(\frac{u,v}{x,y}\right) = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = 0$$

If the Jacobian of u, v is not equal to zero then those functions u, v are functionally independent.

** **Maximum & Minimum for function of a single Variable:**

To find the Maxima & Minima of f(x) we use the following procedure.

(i) Find $f'(x)$ and equate it to zero

(ii) solve the above equation we get x_0, x_1 as roots.

(iii) Then find $f^{11}(x)$.

If $f^{11}(x)_{(x = x0)} > 0$, then f(x) is minimum at x_0

If $f^{11}(x)_{(x = x0)}, < 0$, f(x) is maximum at x_0. Similarly we do this for other stationary points.

PROBLEMS:

1. **Find the max & min of the function $f(x) = x^5 - 3x^4 + 5$ ('08 S-1)**

Sol : Given $f(x) = x^5 - 3x^4 + 5$

$f^1(x) = 5x^4 - 12x^3$

for maxima or minima $f^1(x) = 0$

$5x^4 - 12x^3 = 0$

X =0 , x= 12/5

$f^{11}(x) = 20x^3 - 36x^2$

At $x = 0 \Rightarrow f^{11}(x) = 0$. So f is neither maximum nor minimum at s = 0

At $x = (12/5)$ $f^{11}(x) = 20(12/5)^3 - 36(12/5)$

$= 144(48-36)/25 = 1728/25 > 0$

So f(x) is minimum at x = 12/5

The minimum value is $f(12/5) = (12/5)^5 - 3(12/5)^4 + 5$

** **Maxima & Minima for functions of two Variables**:

Working procedure:

1. Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ Equate each to zero. Solve these equations for x & y we get the pair of values (a_1,b_1) (a_2,b_2) (a_3,b_3) ………………

2. Find $l = \frac{\partial^2 f}{\partial x^2}$, $m = \frac{\partial^2 f}{\partial x \partial y}$, $n = \frac{\partial^2 f}{\partial y^2}$

i) IF $ln - m^2 > 0$ and $l < 0$ at (a_1,b_1) then f(x ,y) is maximum at (a_1,b_1) and maximum value is $f(a_1,b_1)$.

ii) IF $ln - m^2 > 0$ and $l > 0$ at (a_1,b_1) then f(x ,y) is minimum at (a_1,b_1) and minimum value is $f(a_1,b_1)$.

iii) IF $ln - m^2 < 0$ and at (a_1,b_1) then f(x ,y) is neither maximum nor minimum at (a_1,b_1). In this case (a_1,b_1) is saddle point.

iv) IF $ln - m^2 = 0$ and at (a_1,b_1) , no conclusion can be drawn about maximum or minimum and needs further investigation. Similarly we do this for other stationary points.

PROBLEMS:

1. **Locate the stationary points & examine their nature of the following functions.**

('07 S -2)

$u = x^4 + y^4 - 2x^2 + 4xy - 2y^2$, $(x > 0, y > 0)$

Sol: Given $u(x, y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2$

For maxima & minima $\frac{\partial u}{\partial x} = 0$, $\frac{\partial u}{\partial y} = 0$

$\frac{\partial u}{\partial x} = 4x^3 - 4x + 4y = 0 \Rightarrow x^3 - x + y = 0$ -------------------> (1)

$\frac{\partial u}{\partial y} = 4y^3 + 4x - 4y = 0 \Rightarrow y^3 + x - y = 0$ -------------------> (2)

Adding (1) & (2) ,

$x^3 + y^3 = 0$

$\Rightarrow = x = -y$ -------------------> (3)

(1) $\Rightarrow x^2 - 2x \Rightarrow x = 0, \sqrt{2}, -\sqrt{2}$

Hence (3) $\Rightarrow y = 0, -\sqrt{2}, \sqrt{2}$

$l = \frac{\partial^2 f}{\partial x^2} = 12x^2 - 4$, $m = \frac{\partial 2u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = 4$ & $n = \frac{\partial 2u}{\partial y2} = 12y^2 - 4$

$ln - m^2 = (12x^2 - 4)(12y^2 - 4) - 16$

At $(-\sqrt{2}, \sqrt{2})$, $ln - m^2 = (24 - 4)(24 - 4) - 16 = (20)(20) - 16 > 0$

The function has minimum value at $(-\sqrt{2}, \sqrt{2})$

At (0,0) , $ln - m^2 = (0-4)(0-4) - 16 = 0$

(0,0) is not a extrem value.

2. Investigate the maxima & minima if any of the function $f(x) = x^3y^2(1-x-y)$.

('08 S – 4)

Sol: Given $f(x) = x^3y^2(1-x-y) = x^3y^2 - x^4y^2 - x^3y^3$

$\frac{\partial f}{\partial x} = 3x^2y^2 - 4x^3y^2 - 3x^2y^3$ $\quad \frac{\partial f}{\partial y} = 2x^3y - 2x^4y - 3x^3y^2$

For maxima & minima $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$

⇨ $3x^2y^2 - 4x^3y^2 - 3x^2y^3 = 0 \Rightarrow x^2y^2(3 - 4x - 3y) = 0$ ---------------> (1)

⇨ $2x^3y - 2x^4y - 3x^3y^2 = 0 \Rightarrow x^3y(2 - 2x - 3y) = 0$ ----------------> (2)

From (1) & (2) $4x + 3y - 3 = 0$ ----------------X2

$2x + 3y - 2 = 0$ -----------------X3

$2x = 1 \Rightarrow x = ½$

$4(½) + 3y - 3 = 0 \Rightarrow 3y = 3 - 2, \; y = (1/3)$

$l = \frac{\partial^2 f}{\partial x^2} = 6xy^2 - 12x^2y^2 - 6xy^3$

$\left(\frac{\partial^2 f}{\partial x^2}\right)_{(1/2,1/3)} = 6(1/2)(1/3)^2 - 12(1/2)^2(1/3)^2 - 6(1/2)(1/3)^3 = 1/3 - 1/3 - 1/9 = -1/9$

$m = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = 6x^2y - 8x^3y - 9x^2y^2$

$\left(\frac{\partial^2 f}{\partial x \partial y}\right)_{(1/2,1/3)} = 6(1/2)^2(1/3) - 8(1/2)^3(1/3) - 9(1/2)^2(1/3)^3 = \frac{6-4-3}{12} = \frac{-1}{12}$

$n = \frac{\partial 2f}{\partial y2} = 2x^3 - 2x^4 - 6x^3y$

$\left(\frac{\partial^2 f}{\partial y^2}\right)_{(1/2,1/3)} = 2(1/2)^3 - 2(1/2)^4 - 6(1/2)^3(1/3) = \frac{1}{4} - \frac{1}{8} - \frac{1}{4} = -\frac{1}{8}$

$ln - m^2 = (-1/9)(-1/8) - (-1/12)^2 = \frac{1}{72} - \frac{1}{144} = \frac{2-1}{144} = \frac{1}{144} > 0$

The function has a maximum value at (1/2 , 1/3)

3. **Find three positive numbers whose sum is 100 and whose product is maximum.**

('08 S-1)

Sol: Let x ,y ,z be three +ve numbers.

Given $x + y + z = 100$

⇨ $Z = 100 - x - y$

Let $f(x,y) = xyz = xy(100 - x - y) = 100xy - x^2y - xy^2$

For maxima or minima $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$

$\frac{\partial f}{\partial x} = 100y - 2xy - y^2 = 0 \Rightarrow y(100 - 2x - y) = 0$ ----------------> (1)

$\frac{\partial f}{\partial y} = 100x - x^2 - 2xy = 0 \Rightarrow x(100 - x - 2y) = 0$ ------------------> (2)

From (1) & (2)

$100 - 2x - y = 0$

$200 - 2x - 4y = 0$

$-100 + 3y = 0 \Rightarrow 3y = 100 \Rightarrow y = 100/3$

$100 - x - (200/3) = 0 \Rightarrow x = 100/3$

$l = \frac{\partial^2 f}{\partial x^2} = -2y$

$\left(\frac{\partial^2 f}{\partial x^2}\right)(100/3, 100/3) = -200/3$

$m = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = 100 - 2x - 2y$

$\left(\frac{\partial^2 f}{\partial x \partial y}\right)(100/3, 100/3) = 100 - (200/3) - (200/3) = -(100/3)$

$n = \frac{\partial^2 f}{\partial y^2} = -2x$

$\left(\frac{\partial^2 f}{\partial y^2}\right)(100/3, 100/3) = -200/3$

$ln - m^2 = (-200/3)(-200/3) - (-100/3)^2 = (100)^2/3$

The function has a maximum value at (100/3 , 100/3)

i.e. at x = 100/3, y = 100/3 $\therefore z = 100 - \frac{100}{3} - \frac{100}{3} = \frac{100}{3}$

The required no. are x = 100/3, y = 100/3, z = 100/3

4. Find the maxima & minima of the function $f(x) = 2(x^2 - y^2) - x^4 + y^4$ ('08 S-3)

Sol: Given $f(x) = 2(x^2 - y^2) - x^4 + y^4 = 2x^2 - 2y^2 - x^4 + y^4$

For maxima & minima $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$

$\frac{\partial f}{\partial x} = 4x - 4x^3 = 0 \Rightarrow 4x(1 - x^2) = 0 \Rightarrow x = 0, x = \pm 1$

$\frac{\partial f}{\partial y}$ = -4y + 4y^3 = 0 => -4y (1-y^2) = 0 =>y = 0, y = ± 1

l = $\frac{\partial 2f}{\partial x2}$ = 4-12x^2

m = $\frac{\partial 2f}{\partial x \partial y}$ = $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$ = 0

n = $\frac{\partial 2f}{\partial y2}$ = -4 +12y^2

we have ln – m^2 = (4-12x^2)(-4 +12y^2) – 0

= -16 +48x^2 +48y^2 -144x^{2}y^2

= 48x^2 +48y^2 -144x^{2}y^2 -16

i) At (0 , ± 1)

ln – m^2 = 0 + 48 - 0 -16 =32 > 0

l = 4-0 = 4 > 0

f has minimum value at (0 , ± 1)

f (x ,y) = 2(x^2 –y^2) –x^4 +y^4

f (0 , ± 1) = 0 – 2 – 0 + 1 = -1

The minimum value is '-1 '.

ii) At (± 1 ,0)

ln – m^2 = 48 + 0 - 0 -16 =32 > 0

l = 4-12 = - 8 < 0

f has maximum value at (± 1 ,0)

f (x ,y) = 2(x^2 –y^2) –x^4 +y^4

f (± 1 , 0) =2 -0 -1 + 0 = 1

The maximum value is '1 '.

iii) At (0,0) , (± 1 , ± 1)

ln – m^2 < 0

l = 4 -12x^2

(0 , 0) & (± 1 , ± 1) are saddle points.

F has no max & min values at (0 , 0) , (± 1 , ± 1).

Assignment

1. Find the maximum value of x,y,z when x + y + z = a .

   [Ans: $\frac{m^m n^n p^p (a^{m+n+p})}{(m+n+p)^{m+n+p}}$]

***Extremum** : A function which have a maximum or minimum or both is called 'extremum'

***Extreme value** :- The maximum value or minimum value or both of a function is Extreme value.

***Stationary points:** - To get stationary points we solve the equations $\frac{\partial f}{\partial x}$ = 0 and $\frac{\partial f}{\partial y}$ = 0 i.e the pairs (a_1, b_1), (a_2, b_2) ………….. Are called Stationary.

*Maxima & Minima for a function with constant condition :Lagrangian Method

Suppose f(x , y , z) = 0 ------------(1)

$\emptyset$(x , y , z) = 0 ------------- (2)

F(x , y , z) = f(x , y , z) + γ $\emptyset$(x , y , z) where γ is called Lagrange's constant.

1. $\frac{\partial F}{\partial x} = 0 \Rightarrow \frac{\partial f}{\partial x} + \gamma \frac{\partial \emptyset}{\partial x} = 0$ --------------- (3)

   $\frac{\partial F}{\partial y} = 0 \Rightarrow \frac{\partial f}{\partial y} + \gamma \frac{\partial \emptyset}{\partial y} = 0$ --------------- (4)

   $\frac{\partial F}{\partial z} = 0 \Rightarrow \frac{\partial f}{\partial z} + \gamma \frac{\partial \emptyset}{\partial z} = 0$ --------------- (5)

2. Solving the equations (2) (3) (4) & (5) we get the stationary point (x, y, z).
3. Substitute the value of x , y , z in equation (1) we get the extremum.

Problem:

1. **Find the minimum value of $x^2 + y^2 + z^2$ given x + y + z =3a ('08 S-2)**

Sol: $u = x^2 + y^2 + z^2$

$\emptyset = x + y + z - 3a = 0$

Using Lagrange's function

F(x , y , z) = u(x , y , z) + γ $\emptyset$(x , y , z)

For maxima or minima

$\frac{\partial F}{\partial x} = \frac{\partial u}{\partial x} + \gamma \frac{\partial \emptyset}{\partial x} = 2x + \gamma = 0$ ------------ (1)

$\frac{\partial F}{\partial y} = \frac{\partial u}{\partial y} + \gamma \frac{\partial \emptyset}{\partial y} = 2y + \gamma = 0$ ------------ (2)

$\frac{\partial F}{\partial z} = \frac{\partial u}{\partial z} + \gamma \frac{\partial \emptyset}{\partial z} = 2z + \gamma = 0$ ------------ (3)

From (1) , (2) & (3)

$\gamma = -2x = -2y = -2z$

$x = y = z$

$\emptyset = x + x + x - 3a = 0$

$x = a$

$x = y = z = a$

Minimum value of u = $a^2 + a^2 + a^2 = 3a^2$

Fill in the blanks-

1. if u=x+y and v=xy then $\frac{\partial(u,v)}{\partial(x,y)}$ = --------------------------

2. if x=e^ucosv , y= $e^u sin$v then $\frac{\partial(x,y)}{\partial(u,v)}$ = ____________________

3. $J\left(\frac{u,v}{u,v}\right)$ =________________

4. if u= $J\left(\frac{u,v}{x,y}\right)$ then $J\left(\frac{x,y}{u,v}\right)$ = ______________________

5. $\frac{\partial(u,v)}{\partial(x,y)}$ x $\frac{\partial(x,y)}{\partial(u,v)}$ = -------------------------

6. Two functions u and v are said to be functionally dependent if $\frac{\partial(u,v)}{\partial(x,y)}$ = _____________

7. If u=$\frac{x}{y}$ and v=$\frac{x+y}{x-y}$ then $J\left(\frac{u,v}{x,y}\right)$ = _________________

8. If u=e^x siny,v=e^x cosy then $\frac{\partial(u,v)}{\partial(x,y)}$ =__________________

9. $J\left(\frac{u,v}{x,y}\right) . J\left(\frac{x,y}{u,v}\right)$= ______________

10. If x=rcosθ,y=rsinθ then $J\left(\frac{x,y}{u,v}\right)$=_________________

11. If u=3x+5y and v=4x-3y then $\frac{\partial(u,v)}{\partial(x,y)}$ =____________

13. If u=$\frac{y}{x}$ and v=xy then $J\left(\frac{u,v}{x,y}\right)$= __________________

OBJECTIVE TYPE QUESTIONS

14. If u=x^2 - 2y ,v=x+y then $\frac{\partial(u,v)}{\partial(x,y)}$ = ________________

(a) $(x+1)^2$ (b) 2(x+1) (c) 3(x+1) (d) None

16. If u(1-v)=x, uv=y then $J\left(\frac{u,v}{x,y}\right)$. $J\left(\frac{x,y}{u,v}\right)$ =

(a) 0 (b) 1 (c) xy (d) None

17. If u=$\frac{x+y}{1-xy}$, v=tan^{-1} x + tan^{-1} y then $J\left(\frac{u,v}{x,y}\right)$. $J\left(\frac{x,y}{u,v}\right)$ =

(a) 0 (b) 1 (c) xy (d) None

18. Are u=x$\sqrt{1-x^2}$, v=2x functionally dependent? If so what is $\left(\frac{u,v}{x,y}\right)$?

(a) yes,1 (b) yes,0 (c) No,0 (d) None

19. If u=x^2y ,v=xy^2 then $\frac{\partial(u,v)}{\partial(x,y)}$ is

(a)5x^2y^2 (b)4 x^2y^2 (c) $2x^2y^2$ (d) $3x^2y^2$

(Assignment Questions)

{ Functions of Several Variables}

1. If x+y^2=u , y+z^2=v ,z+ x^2=w find $\frac{\partial(x,y,z)}{\partial(u,v,w)}$.
2. If x+y+z=u, y+z= uv, z=uvw then evaluate $\frac{\partial(x,y,z)}{\partial(u,v,w)}$.
3. S.T the functions u=x+y+z, v=$x^2+y^2+z^2$-2xy-2zx and w=$x^3+y^3+z^3$-3xyz are functionally related.
4. Find the max & min values of the function f(x)=x^5-3x^4+5.
5. Find three positive numbers whose sum is 100 and whose product is maximum.
6. Locate the stationary points & examine their nature of the following functions u=$x^4+y^4-2x^2+4xy-2y^2$ (x>0,y>0).
7. If u= $\frac{yz}{x}$,v=$\frac{xz}{y}$, w=$\frac{xy}{z}$, find $\frac{\partial(u,v,w)}{\partial(x,y,z)}$.